Please provide complete and well-written solutions to the following exercises.

Due February 15, at the beginning of class.

Homework 4

Exercise 1. Let $\varepsilon_1, \varepsilon_2, \cdots \in \{0,1\}$ be random variables that are independent and identically distributed copies of the Bernoulli random variable with expectation $1/2$, so that $\mathbf{P}(\varepsilon_n = 1) = \mathbf{P}(\varepsilon_n = 0) = 1/2$ for all $n \geq 1$.

- Show that the random variable $\sum_{n=1}^{\infty} 2^{-n}\varepsilon_n$ is uniformly distributed on the unit interval $[0,1]$.
- Show that the random variable $\sum_{n=1}^{\infty} 2{\cdot}3^{-n}\varepsilon_n$ is uniformly distributed on the standard middle third Cantor set (where the Cantor set's center is $1/2$.)
- Let μ be a probability measure on $\mathbf{R}$. The Fourier Transform of μ at $\xi \in \mathbf{R}$ is defined by $\widehat{\mu}(\xi) := \int_{\mathbf{R}} e^{ix\xi} d\mu(x)$. where $i = \sqrt{-1}$. For example, if μ is uniform on $[-1/2, 1/2]$, then

$$\widehat{\mu}(\xi) = \int_{-1/2}^{1/2} e^{ix\xi} dx = \frac{e^{i\xi/2} - e^{-i\xi/2}}{i\xi} = \frac{2\sin(\xi/2)}{\xi}, \qquad \forall\, \xi \neq 0.$$

  Using the first item, find an expression for $\sin(\xi)/\xi$ in terms of an infinite product of cosines. (Hint: if a random variable X has distribution μ_X, then $\widehat{\mu_X}(\xi) = \mathbf{E}e^{iX\xi}$ for any $\xi \in \mathbf{R}$. So the Fourier transform of the sum of independent random variables is the product of the Fourier transforms.) Similarly, find an expression for the Fourier transform of the uniform measure on the middle third Cantor set (when the Cantor set's center is $0 \in \mathbf{R}$) in terms of an infinite product of cosines.

Exercise 2. Let X be a random variable taking nonnegative integer values. Show that

$$\mathbf{E}X = \sum_{n=1}^{\infty} \mathbf{P}(X \geq n).$$

Exercise 3 (MAX-CUT)**.** The probabilistic method is a very useful way to prove the existence of something satisfying some properties. This method is based upon the following elementary statement: If $\alpha \in \mathbf{R}$ and if a random variable $X \colon \Omega \to \mathbf{R}$ satisfies $\mathbf{E}X \geq \alpha$, then there exists some $\omega \in \Omega$ such that $X(\omega) \geq \alpha$. We will demonstrate this principle in this exercise.

Let $G = (V, E)$ be an undirected graph on the vertices $V = \{1, \ldots, n\}$ so that the edge set E is a subset of unordered pairs $\{i, j\}$ such that $i, j \in V$ and $i \neq j$. Let $S \subseteq V$ and denote $S^c := V \smallsetminus S$. We refer to (S, S^c) as a cut of the graph G. The goal of the MAX-CUT problem is to maximize the number of edges going between S and S^c over all cuts of the graph G.

Prove that there exists a cut (S, S^c) of the graph such that the number of edges going between S and S^c is at least $|E|/2$. (Hint: define a random $S \subseteq V$ such that, for every $i \in V$, $\mathbf{P}(i \in S) = 1/2$, and the events $1 \in S, 2 \in S, \ldots, n \in S$ are all independent. If $\{i, j\} \in E$, show that $\mathbf{P}(i \in S, j \notin S) = 1/2$. So, what is the expected number of edges $\{i, j\} \in E$ such that $i \in S$ and $j \notin S$?)

Exercise 4. Let $X_1, X_2, \ldots : \Omega \to S$ be random variables. Show that

$$\sigma(X_1, X_2, \ldots) = \sigma(\cup_{i=1}^{\infty} \sigma(X_1, \ldots, X_i)).$$

Exercise 5. Let $(X_i)_{i \in I}$ be a collection of independent random variables. Show that $(X_i)_{i \in I}$ are independent if and only if $(\sigma(X_i))_{i \in I}$ are independent σ-algebras. (Hint: Let $i \in I$ and let $J \subseteq I \setminus \{i\}$ be finite. Are the sets in $\sigma(X_i)$ that are independent of $(\sigma(X_j))_{j \in J}$ a monotone class?)

Exercise 6. Let $X_1, X_2, \ldots$ be random variables. Show that $X_1, X_2, \ldots$ are independent if and only if: for every $i \geq 1$, $\sigma(X_{i+1})$ is independent of $\sigma(X_1, \ldots, X_i)$. And the previous cases occur if and only if: for every $i \geq 1$, $\sigma(X_{i+1}, X_{i+1}, \ldots)$ is independent of $\sigma(X_1, \ldots, X_i)$

Exercise 7. Let $X_1, X_2, \ldots : \Omega \to \mathbf{R}$ be a sequence of independent random variables. For any $n \geq 1$, let $S_n := X_1 + \cdots + X_n$. Show the following:

- $\{\lim_{n \to \infty} S_n \text{ exists}\} \in \mathcal{T}$.
- If $t \in [-\infty, \infty]$, then it can occur that $\{\limsup_{n \to \infty} S_n > t\} \notin \mathcal{T}$.
- If $t \in [-\infty, \infty]$ and if $c_1 \leq c_2 \leq \cdots$ is a sequence of real numbers such that $\lim_{n \to \infty} c_n = \infty$, then
$$\{\limsup_{n \to \infty} \frac{S_n}{c_n} > t\} \in \mathcal{T}.$$