1: INTRODUCTION, NATURAL NUMBERS, REAL NUMBERS

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Abstract. These notes are mostly copied from those of T. Tao from 2003, available here

Contents

1. Introductory Remarks

1.1. **A rigorous version of calculus.** Here is a "proof" of Euler which in 1735 found the quantity $1 + 1/4 + 1/9 + 1/16 + \cdots$, thereby solving the Basel problem. Do you agree with the logic? Let x be a real number. Then

$$
\begin{aligned}
1 - \pi^2 x^2/6 + \cdots &= \frac{\sin(\pi x)}{\pi x} \quad \text{, by Taylor series} && (1)\\
&= (1-x)(1+x)(1-x/2)(1+x/2)(1-x/3)(1+x/3)\cdots && (2)\\
&\qquad \text{, since a nice function is a product of its zeros} \\
&= (1-x^2)(1-x^2/4)(1-x^2/9)\cdots && (3)\\
&= 1 - x^2(1 + 1/4 + 1/9 + \cdots) + x^4(\cdots) + \cdots && (4)
\end{aligned}
$$

So, equation the x^2 terms on both sides, we get

$$1 + 1/4 + 1/9 + 1/16 + \cdots = \pi^2/6. \qquad (5)$$

It is actually possible to make this argument rigorous, but what problems do you see with the amount of rigor? I see a few:

- In what sense does equality hold in (1)?
- What is the true meaning of an infinite sum, as in (1)?
- What is the meaning of the infinite product in (2)?

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- Is every function really the product of its zeros? This seems quite unlikely. (In fact it is false in general (consider e^x), but (2) actually does hold in an appropriate sense.)
- Can we freely rearrange terms in an infinite sum or an infinite product as in (3) and (4)? (In general, we cannot, but sometimes we can.)

Euler was a brilliant mathematician, but he also occasionally made some mistakes by using non-rigorous methods. Using intuition and non-rigorous calculations can be very helpful, though! No one else was able to find (5) at the time. Yet, in order to be entirely certain of facts, we need to ultimately find rigorous proofs of these facts. The above proof would receive only partial credit as a solution on a homework, since it is no longer 1735.

1.2. **What will we be learning?** We will learn a fully rigorous version of calculus. That is, we will learn how to answer many of the questions raised in the previous section. The ultimate goal of the course is to develop an ability to read and write rigorous proofs of mathematics. Also, we would like to learn how to rigorously treat calculus. From the time of Newton and Leibniz in the mid 1600s to the time of Cauchy in the mid 1800s, calculus did not have a truly rigorous foundation. And developing such a foundation turned out to be a fairly difficult problem, which arguably lasted to the time of Cantor in the early 1900s. Such a rigorous foundation has been quite influential in all other areas of mathematics.

More generally, in nearly any vocation or avocation, the process of problem solving and thinking rigorously that we learn in this class can be applicable. There is a reason that Euclid's *Elements* were learned by many students in the past, and there is a reason that this abstract, axiomatic method is still taught in our mathematics classes today.

1.3. **How will we be learning analysis?** As in the Euclidean axiomatization of geometry, we will begin with the most basic axioms of arithmetic, and we will slowly build up our understanding of numbers. For example, one question that we did not yet address is:

What is a real number?

We perhaps have a good intuitive idea of what a real number is. But what is a real number, really? Maybe you think of a real number in terms of some infinite decimal. So, are the real numbers the set of infinite decimals? For example,

$$1.000000\ldots$$

$$3.141592653589\ldots$$

$$1.34300344300\ldots$$

This seems reasonable at first, but there are some issues with this definition. For example, the following two decimals should really be the same number, even though they look very different.

$$1.00000\ldots \qquad \text{and} \qquad 0.9999999\ldots$$

If you don't agree that these are the same number, then consider what their difference is.

By adjusting for this issue, it is possible to define the real numbers in terms of infinite decimals. However, there are other, better definitions of the real numbers, which are more instructive and more useful later. We will construct the real numbers soon using so-called Cauchy sequences. In order to adjust to axiomatic thinking, and to review induction, we start at the very beginning and define the natural numbers. We emphasize at the outset that we will treat numbers as abstract mathematical objects that satisfy certain properties.

Such a treatment perhaps lacks some intuition, but it seems necessary to provide a rigorous foundation of mathematics that can avoid some of the issues we discussed in Euler's proof above. On the other hand, intuition can be quite useful in proving various facts. So, doing mathematics seems to require two complementary modes of thought: the nonrigorous, creative mode, and the rigorous, logical mode.

In this first chapter, we will begin with the axiomatization of the natural numbers, and we will then move to axiomatizations of the integers, rationals, and reals, respectively. The point of studying the axiomatization of the natural numbers is that it will allow a review of induction, and it will lead naturally to our eventual axiomatization of the real number system. However, a rigorous axiomatization of the real number system is a surprisingly difficult creation.

1.4. **Why are we learning this material?** This material lays the foundation for a great deal of further subjects. To give just one example, consider Fourier analysis, which is arguably one of the most seminal areas of mathematics. Every time we use a cell phone, or look at a JPEG, or watch an online video (for example, an MPEG), or when a doctor uses an MRI or CT-Scan, Fourier analysis is involved. In Fourier analysis, we begin with a function, we break this function up into simpler pieces, and we then reassemble these pieces. Sometimes we are allowed to break up the function into pieces, and sometimes we are not. The details become unexpectedly subtle. The rigorous way of thinking and the results of this course play a crucial role in dealing with the details of the subject of Fourier analysis.

Abstract reasoning has some advantages and disadvantages. Since abstract reasoning usually does not come naturally, it can be difficult to learn material that is presented in an abstract way. On the other hand, an abstract approach promises more applicability. For example, there are many different ways to interpret a real-valued function on the real line. Such a function could represent the amplitude of a sound wave over time, the price of a stock over time, the displacement of an object over time, and so on.

2. Natural Numbers

The natural numbers $\mathbb{N}$ are defined by the following axioms.

Definition 2.1 (**Peano Axioms**).

(1) 0 is a natural number.
(2) Every natural number n has a successor $n++$ which is also a natural number.
(3) 0 is not the successor of any natural number. That is, for any natural number n, $n{+}{+} \neq 0$.
(4) Different natural numbers have difference successors. That is, if n, m are natural numbers with $n \neq m$, then $n++ \neq m++$.
(5) (**Principle of Induction**) Let n be a natural number, and let $P(n)$ be any property that holds for n. Assume that $P(0)$ is true, and whenever $P(n)$ is true for any natural number n, $P(n++)$ is also true. Then $P(n)$ is true for every natural number n.

Assumption 1 (**The Natural Numbers**). There exists a number system $\mathbb{N}$, whose elements we call **natural numbers**, such that Axioms (1) through (5) of Definition 2.1 are true.

Definition 2.2. Define $1 := 0++$.

Definition 2.3 (**Addition of Natural Numbers**)**.** Let m be a natural number. Define $0+m := m$. We now define how to add other natural numbers to m. Let n be a natural number. Suppose we have inductively defined $n+m$. Then, define $(n{+}{+})+m := (n+m){+}{+}$.

Remark 2.4. By Axiom (5), we have defined addition on all natural numbers n, m.

Exercise 2.5. Show that, from Axioms $(1),(2)$ it follows by induction (using Axiom (5)) that addition of two natural numbers produces a natural number.

Remark 2.6. Using only the definitions $0+m=m$ and $(n++)+m=(n+m)++$, we will deduce all basic facts of arithmetic.

Lemma 2.7. *For any natural number n, $n+0=n$.*

Remark 2.8. Note that we cannot apply commutativity of addition, since it does not immediately follow from the axioms of Definition 2.1.

Proof. From Definition 2.3, $0+0=0$. So, we induct on n. Suppose $n+0=n$ for a natural number n. We need to show that $(n++)+0=n++$. From Definition 2.3, $(n{+}{+})+0=(n+0){+}{+}$. From the inductive hypothesis, we therefore have $(n{+}{+})+0=n{+}{+}$, as desired. Having completed the inductive step and the base case, we are done. □

Lemma 2.9. *For any natural numbers n, m, we have $n+(m++)=(n+m)++$*

Proof. We fix m and induct on n. In the base case $n=0$, we need to show $0+(m++)=(0+m)++$. From Definition 2.3, we know that $0+(m{+}{+})=m{+}{+}$ and $(0+m){+}{+}=m{+}{+}$. We conclude that $0+(m++)=(0+m)++$, as desired. We now induct on n. Suppose n satisfies $n+(m++)=(n+m)++$. We need to show that

$$(n++)+(m++)=((n++)+m)++. \qquad (*)$$

From Definition 2.3, $(n++)+(m++)=(n+(m++))++$. From the inductive hypothesis, $(n+(m++))++=((n+m)++)++$. From Definition 2.3, $((n++)+m)++=((n+m)++)++$. We conclude that both sides of $(*)$ are equal, so the inductive step holds, and we deduce the lemma. □

Remark 2.10. From Definition 2.2, Lemma 2.7 and Lemma 2.9, $n+1=n+(0++)=(n+0)++=n++$, so $n++=n+1$ for all natural numbers n.

Proposition 2.11 (**Addition is Commutative**)**.** *For any natural numbers n, m, we have $n+m=m+n$.*

Proof. We fix m and induct on n. In the base case $n=0$, we need to show that $0{+}m=m{+}0$. From Definition 2.3, $0+m=m$. From Lemma 2.7, $m+0=m$. Therefore, $0+m=m+0$, as desired. Now, assume that $n+m=m+n$. We need to show that

$$(n++)+m=m+(n++). \qquad (*)$$

From Definition 2.3, $(n{+}{+})+m=(n+m){+}{+}$. From Lemma 2.9, $m+(n{+}{+})=(m+n){+}{+}$. From the inductive hypothesis, $(m+n)++=(n+m)++$. Putting everything together $(*)$ holds, and the inductive step is complete. □

Proposition 2.12 (**Addition is Associative**)**.** *For any natural numbers a, b, c, we have $(a+b)+c=a+(b+c)$.*

Exercise 2.13. Prove Proposition 2.12 by fixing two variables and inducting on the third variable.

Proposition 2.14 (**Cancellation Law**)**.** *Let a, b, c be natural numbers such that $a + b = a + c$. Then $b = c$.*

Remark 2.15. We have not defined subtraction, so we cannot subtract a from both sides. In fact, we will use the Cancellation Law to *define* subtraction.

Proof. We induct on a. For the base case $a = 0$, we assume that $0 + b = 0 + c$. From Definition 2.3, we conclude that $b = c$, thereby proving the base case. Now, assume that: if $a + b = a + c$, then $b = c$. We need to show that: if $(a + +) + b = (a + +) + c$, then $b = c$. From Definition 2.3, $(a + +) + b = (a + b) + +$. Similarly, $(a + +) + c = (a + c) + +$. So, we know that $(a + b) + + = (a + c) + +$. From the contrapositive of Axiom (4) of Definition 2.1, we conclude that $a + b = a + c$. From the inductive hypothesis, $b = c$. So, the inductive step is complete, and we are done. $\square$

Definition 2.16 (**Positivity**)**.** A natural number n is said to be **positive** if and only if $n \neq 0$.

Proposition 2.17. *Let a, b be natural numbers. Assume that a is positive. Then $a + b$ is positive.*

Proof. We induct on b. For the base case, $b = 0$, and we see that $a+b = a+0 = a$. Since a is positive, we conclude that $a + b$ is positive. We now prove the inductive step. Assume that $a+b$ is positive. We need to show that $a+(b++)$ is positive. But $a+(b++) = (a+b)++$, and $(a + b) + + \neq 0$ by Axiom (3) of Definition 2.1. We have therefore completed the inductive step. $\square$

The following Corollary is the contrapositive of Proposition 2.17.

Corollary 2.18. *Let a, b be natural numbers such that $a + b = 0$. Then $a = b = 0$.*

Definition 2.19 (**Order**)**.** Let n, m be natural numbers. We say that n **is greater than or equal to** m, and we write $n \geq m$ or $m \leq n$, if and only if $n = m + a$ for some natural number a. We say that n **is strictly greater than** m, and we write $n > m$ or $m < n$, if and only if $n \geq m$ and $n \neq m$.

Proposition 2.20 (**Properties of Order**)**.** *Let a, b, c be natural numbers.*

(1) *$a \geq a$.*
(2) *If $a \geq b$ and $b \geq c$, then $a \geq c$.*
(3) *If $a \geq b$ and $b \geq a$, then $a = b$.*
(4) *$a \geq b$ if and only if $a + c \geq b + c$.*
(5) *$a < b$ if and only if $a + c < b + c$.*

Exercise 2.21. Prove Proposition 2.20.

Proposition 2.22 (**Trichotomy of Order**)**.** *Let a, b be natural numbers. Then exactly one of the following statements is true: $a < b$, $a > b$ or $a = b$.*

2.1. **Multiplication.**

Remark 2.23. We will now freely use facts about addition of natural numbers, without referencing the above lemmas and propositions.

Definition 2.24 (Multiplication). Let m be a natural number. We define multiplication $\times$ as follows. Define $0 \times m := 0$. Now, let n be a natural number, and assume we have inductively defined $n \times m$. Then, define $(n++) \times m := (n \times m) + m$.

Remark 2.25. One can show by induction that $n \times m$ is a natural number, for any natural numbers n, m.

Exercise 2.26. Imitating the proofs of Lemmas 2.7 and 2.9 and Proposition 2.11, show that, for all natural numbers n, m, we have $n \times 0 = 0$, $n \times (m++) = (n \times m) + n$ and $n \times m = m \times n$.

Remark 2.27. Let n, m, r be natural numbers. As is standard, we write nm to denote $n \times m$. Also, $nm + r$ denotes $(n \times m) + r$.

Remark 2.28. If a, b are positive natural numbers, than ab is positive. One can prove this using induction and Proposition 2.17.

Proposition 2.29 (Distributive Law). *For any natural numbers a, b, c, we have $a(b+c) = ab + ac$ and $(b + c)a = ba + ca$.*

Proof. From Exercise 2.26, multiplication is commutative. So, it suffices to prove $a(b + c) = ab + ac$. Fix a, b. We then induct on c. The base case corresponds to $c = 0$. We need to prove $a(b + 0) = ab + a0$. The left side is ab, and the right side is $ab + 0 = ab$, so the base case is verified. Now, assume that $a(b + c) = ab + ac$ for some natural number c. We need to show that $a(b + (c++)) = ab + a(c++)$. The left side is $a((b + c)++) = a(b + c) + a$, by Definition 2.24. So, by the inductive hypothesis, the left side is $ab + ac + a$. Meanwhile, the right side is $ab + ac + a$, by Definition 2.24. So, the inductive step has been completed. □

Remark 2.30. From Proposition 2.29, we can mimic the proof of Proposition 2.12 to prove that, for all natural numbers a, b, c, we have $a(bc) = (ab)c$.

Proposition 2.31. *Let a, b be natural numbers with $a < b$. If c is a positive natural number, then $ac < bc$.*

Proof. Since $a < b$, there exists a positive natural number d such that $a + d = b$. Multiplying both sides by c and using Proposition 2.29, $bc = ac + dc$. Since d, c are positive, dc is positive by Remark 2.28. We conclude that $ac < bc$ by the definition of order, as desired. □

Corollary 2.32 (Cancellation Law). *Let a, b, c be natural numbers such that $ac = bc$ and such that $c \neq 0$. Then $a = b$.*

Proof. From the trichotomy of order (Proposition 2.22), either $a < b$, $a > b$ or $a = b$. Since $c \neq 0$, c is positive. So, if $a < b$, then $ac < bc$ by Proposition 2.31. Similarly, if $b < a$, then $bc < ac$ by Proposition 2.31. So, the cases $a < b$ and $b < a$ cannot occur. We conclude that $a = b$, as desired. □

Remark 2.33. From now on, we will write $n++$ as $n + 1$, and we will use basic properties of addition and multiplication of natural numbers.

Proposition 2.34 (The Euclidean Algorithm). *Let n be a natural number and let q be a positive natural number. Then there exist natural numbers m, r such that $0 \leq r < q$ and such that $n = mq + r$.*

Remark 2.35. That is, we can divide n by q, leaving a remainder r, where $0 \leq r < q$.

Exercise 2.36. Prove Proposition 2.34 by fixing q and using induction on n.

3. Integers

We have dealt with addition and multiplication of natural numbers above. We would now like to deal with subtraction. In order to do this, we need to construct the integers. We will define the integers as the formal difference of two natural numbers. This is not the only way to define the integers, but it ends up being a bit cleaner than other methods.

Definition 3.1 (Integers). An **integer** is an expression of the form a——b where a, b are natural numbers. We say that two integers a——b and c——d are equal if and only if $a + d = c + b$. We let $\mathbb{Z}$ denote the set of all integers.

Example 3.2. So, the integer 5——2 is equal to 4——1 since $5 + 1 = 4 + 2$.

Remark 3.3. We need to verify that three axioms hold for this notion of equality. For any natural numbers a, b, c, d, e, f, we need to show:

1. a——b is equal to a——b.
2. If a——b is equal to c——d, then c——d is equal to a——b.
3. If a——b is equal to c——d, and if c——d is equal to e——f, then a——b is equal to e——f.

These three axioms define an equivalence relation on integers. Properties (1) and (2) follow immediately. To show property (3), note that $a + d = b + c$, and $c + f = d + e$. Adding both equations, we get $a + d + c + f = b + c + d + e$. From the Cancellation Law (Proposition 2.14), we conclude that $a + f = b + e$, so that a——b is equal to e——f, as desired.

Definition 3.4 (Addition and Multiplication of Integers). Let a——b and c——d be two integers. We define the sum $(a$——$b) + (c$——$d)$ by

$$(a\text{——}b) + (c\text{——}d) := (a + c)\text{——}(b + d).$$

We define the product $(a$——$b) \times (c$——$d)$ by

$$(a\text{——}b) \times (c\text{——}d) := (ac + bd)\text{——}(ad + bc).$$

One potential problem with these definitions is that, even though 5——2=4——1, it is not clear that $(5$——$2) + (c$——$d) = (4$——$1) + (c$——$d)$, or that $(5$——$2) \times (c$——$d) = (4$——$1) \times (c$——$d)$. Fortunately, this is not a problem at all.

Lemma 3.5. *Let a, a', b, b', c, d be natural numbers such that a——b=a'——b'. Then*

1. $(a\text{——}b) + (c\text{——}d) = (a'\text{——}b') + (c\text{——}d)$.
2. $(a\text{——}b) \times (c\text{——}d) = (a'\text{——}b') \times (c\text{——}d)$.
3. $(c\text{——}d) + (a\text{——}b) = (c\text{——}d) + (a'\text{——}b')$.
4. $(c\text{——}d) \times (a\text{——}b) = (c\text{——}d) \times (a'\text{——}b')$.

Proof. We first prove (1). Using Definition 3.4, we need to show that $(a+c)$——$(b+d) = (a'+c)$——$(b'+d)$. Using Definition 3.1, we need to show that $a+c+b'+d = a'+c+b+d$. Since a——b=a'——b', we know that $a+b' = a'+b$. So, adding $c+d$ to both sides proves (1).

We now prove (2). Using Definition 3.4, we need to show that $(ac+bd)$——$(bc+ad) = (a'c+b'd)$——$(b'c+a'd)$. Using Definition 3.1, we need to show that $ac+bd+b'c+a'd = a'c+b'd+bc+ad$. The left side can be written $c(a+b')+d(a'+b)$, while the right is $c(a'+b)+d(a+b')$. Since a——b=a'——b', we know that $a+b' = a'+b$. So, both sides of (2) are equal. The remaining claims $(3), (4)$ are proven similarly. □

Remark 3.6. Let n, m be any natural numbers. Then the set of integers n——0 behave exactly like the natural numbers. For example, $(n$——$0) + (m$——$0) = (n+m)$——0, and $(n$——$0) \times (m$——$0) = (nm)$——0. Also, $(n$——$0) = (m$——$0)$ if and only if $n = m$. So, we may identify the natural numbers as a subset of the integers via the correspondence $n = (n$——$0)$. Note in particular that under this correspondence, $0 = (0$——$0)$ and $1 = (1$——$0)$.

Remark 3.7. Then, for any integer x, we define $x{+}{+} := x+1$.

Definition 3.8. Let $(a$——$b)$ be an integer. We define the **negation** $-(a$——$b)$ of $(a$——$b)$ by $-(a$——$b) := (b$——$a)$.

Remark 3.9. Negation is well-defined. That is, if $(a$——$b) = (a'$——$b')$, then $-(a$——$b) = -(a$——$b)$.

Definition 3.10. Let n be a natural number. We define $-n := -(n$——$0) = (0$——$n)$. If n is a positive natural number, we call $-n$ a **negative integer**.

Lemma 3.11. *Let x be an integer. Then exactly one of the following three statements is true.*

(1) *x is zero.*
(2) *There exists a positive natural number n such that $x = n$.*
(3) *There exists a positive natural number n such that $x = -n$.*

Proposition 3.12. *Let x, y, z be integers. Then the following laws of algebra hold.*

- $x+y = y+x$ *(Commutativity of addition)*
- $(x+y)+z = x+(y+z)$ *(Associativity of addition)*
- $x+0 = 0+x = x$ *(Additive identity element)*
- $x+(-x) = (-x)+x = 0$ *(Additive inverse)*
- $xy = yx$ *(Commutativity of multiplication)*
- $(xy)z = x(yz)$ *(Associativity of multiplication)*
- $x1 = 1x = x$ *(Multiplicative identity element)*
- $x(y+z) = xy+xz$ *(Left Distributivity)*
- $(y+z)x = yx+zx$ *(Right Distributivity)*

Remark 3.13. These properties say that the integers form a **commutative ring**. Note that there is no notion of division within the integers. More specifically, there is no multiplicative inverse property. For example, given $2 \in \mathbb{Z}$, there does not exist an $x \in Z$ such that $2x = 1$. In order to have multiplicative inverses, we will need to enlarge the set of integers to the set of rational numbers. We will realize this goal shortly.

Proof of Associativity of addition. Let x, y, z be integers. Then there exist natural numbers a, b, c, d, e, f such that $x = a{-\!\!-}b$, $y = c{-\!\!-}d$ and such that $z = e{-\!\!-}f$. We compute both sides of the purported inequality $(xy)z = x(yz)$, separately.

$$\begin{aligned}(xy)z &= [(a{-\!\!-}b)(c{-\!\!-}d)](e{-\!\!-}f) = [(ac+bd){-\!\!-}(bc+ad)](e{-\!\!-}f)\\ &= (ace+bde+bcf+adf){-\!\!-}(acf+bdf+bce+ade).\end{aligned}$$

$$\begin{aligned}x(yz) &= (a{-\!\!-}b)[(c{-\!\!-}d)(e{-\!\!-}f)] = (a{-\!\!-}b)[(ce+df){-\!\!-}(cf+de)]\\ &= (ace+adf+bcf+bde){-\!\!-}(bce+bdf+acf+ade).\end{aligned}$$

So, $(xy)z = x(yz)$ for all integers x, y, z, as desired. □

Proposition 3.14. *Let a, b be integers such that $ab = 0$. Then at least one of a, b is zero.*

Exercise 3.15. Prove Proposition 3.14.

Corollary 3.16 (Cancellation Law). *Let a, b, c be integers such that $c \neq 0$ and such that $ac = bc$. Then $a = b$.*

Proof. Since $ac = bc$, we have $(a - b)c = ac - bc = 0$. Since $c \neq 0$, Proposition 3.14 implies that $a - b = 0$, so that $a = b$. □

We can now define the order on the integers exactly as we did for the natural numbers.

Definition 3.17 (Order). Let n, m be integers. We say that n **is greater than or equal to** m, and we write $n \geq m$ or $m \leq n$, if and only if $n = m + a$ for some natural number a. We say that n **is strictly greater than** m, and we write $n > m$ or $m < n$, if and only if $n \geq m$ and $n \neq m$.

Also, using Proposition 3.12, we have the following properties of order

Proposition 3.18 (Properties of Order). *Let a, b be integers.*

(1) *$a > b$ if and only if $a - b$ is a positive natural number.*
(2) *If $a > b$, then $a + c > b + c$ for any integer c.*
(3) *If $a > b$, then $ac > bc$ for any positive natural number c.*
(4) *If $a > b$, then $-a < -b$.*
(5) *If $a > b$ and $b > c$, then $a > c$.*
(6) *If $a \geq b$ and $b \geq a$, then $a = b$.*

4. Rationals

As discussed above, there does not exist an integer x such that $2x = 1$. That is, a general integer does not have a multiplicative inverse. In order to get multiplicative inverses for nonzero integers, we need to enlarge this set to the set of rational numbers. As above, we will define the rational numbers axiomatically.

Definition 4.1 (Rational Numbers). A **rational number** is an expression of the form $a//b$, where a, b are integers and $b \neq 0$. Two rational numbers $a//b$ and $c//d$ are considered to be equal if and only if $ad = cb$.

Remark 4.2. As before, we need to check that this notion of equality of rational numbers is an equivalence relation. It follows readily that $a//b$ is equal to $a//b$, and if $a//b$ is equal to $c//d$, then $c//d$ is equal to $a//b$. To check the third property, suppose $a//b$ is equal to $c//d$, and $c//d$ is equal to $e//f$. Then $ad = bc$ and $cf = de$. Multiplying both of these equations, we get $adcf = debc$. We need to show that $a//b$ is equal to $e//f$. That is, we need to show that $af = eb$. Since $d \neq 0$, from the Cancellation Law (Corollary 3.16), the equation $adcf = debc$ becomes $acf = ebc$. If $c \neq 0$, the Cancellation law implies that $af = eb$, as desired. If $c = 0$, then $ad = bc = 0$ and $de = cf = 0$. And since $b \neq 0$ and $d \neq 0$, Proposition 3.14 implies that $a = e = 0$. So, $af = 0 = eb$, as desired. In any case, we have proven that our notion of equality of rational numbers is an equivalence relation.

As before, we now define addition, multiplication, and negation of rational numbers. And we then need to check that these definitions are well-defined.

Definition 4.3. Let $a//b$ and $c//d$ be rational numbers. Define their **sum** as follows.

$$(a//b) + (c//d) = (ad + bc)//(db).$$

Define their **product** as follows.

$$(a//b) \times (c//d) := (ac)//(bd).$$

Define the **negation** of $a//b$ as follows.

$$-(a//b) := (-a)//b.$$

Lemma 4.4. *Let $a//b, a'//b', c//d$ be rational numbers such that $a//b$ is equal to $a'//b'$. Then the sum, product, and negation are unchanged when we replace $a//b$ with $a'//b'$. And similarly for $c//d$.*

Proof. We prove the first property, since the other proofs are similar. We need to show that $(a//b) + (c//d) = (a'//b') + (c//d)$. That is, we need to show that $(ad + bc)//(bd) = (a'd + b'c)//(b'd)$. That is, we need to show that $(ad + bc)(b'd) = (a'd + b'c)(bd)$, i.e. we need $ab'dd + bb'cd = a'bdd + bb'cd$, i.e. we need $ab'dd = a'bdd$. We know that $a//b = a'//b'$. That is, we know that $ab' = a'b$. So, the claim follows by multiplying both sides of this equation by dd, as desired. □

Remark 4.5. Let a, b be integers. The rational numbers $a//1, b//1$ behave exactly like the integers, since we have

$$(a//1) + (b//1) = (a + b)//1, \qquad (a//1) \times (b//1) = (ab)//1, \qquad -(a//1) = (-a)//1.$$

Also, $a//1 = b//1$ if and only if $a = b$. We therefore identify the rational numbers $a//1$ with the integers a by the relation $a = a//1$.

Remark 4.6. Let $a//b$ be a rational number. Then $a//b = 0//1$ if and only if $a = 0$. Taking the contrapositive, $a//b \neq 0//1$ if and only if $a \neq 0$.

Definition 4.7 (Reciprocal). Let $x = a//b$ be a nonzero rational number. From the previous remark and the definition of rational numbers, $a \neq 0$ and $b \neq 0$. We then define the **reciprocal** x^{-1} of x by $x^{-1} := b//a$. Note that if two rational numbers are equal, then their reciprocals are equal. Also, the reciprocal of 0 is left undefined.

Just as in the case of the integers, we can now prove various properties of the rationals. However, as promised, we now have an additional property. Nonzero numbers now have a multiplicative inverse. Whereas the integers were a commutative ring, the rationals are also a commutative ring. And with this additional multiplicative inverse property, the rationals are now referred to as a **field**.

Proposition 4.8. *Let x, y, z be rational numbers. Then the following laws of algebra hold.*

- $x + y = y + x$ *(Commutativity of addition)*
- $(x + y) + z = x + (y + z)$ *(Associativity of addition)*
- $x + 0 = 0 + x = x$ *(Additive identity element)*
- $x + (-x) = (-x) + x = 0$ *(Additive inverse)*
- $xy = yx$ *(Commutativity of multiplication)*
- $(xy)z = x(yz)$ *(Associativity of multiplication)*
- $x1 = 1x = x$ *(Multiplicative identity element)*
- $x(y + z) = xy + xz$ *(Left Distributivity)*
- $(y + z)x = yx + zx$ *(Right Distributivity)*

Finally, if x is nonzero, then

- $xx^{-1} = x^{-1}x = 1$ *(Multiplicative Inverse)*

Proof. We will only prove the associativity of addition, since the other proofs have a similar flavor. Write $x = a//b$, $y = c//d$, $z = e//f$. Then

$$\begin{aligned}(x + y) + z &= ((a//b) + (c//d)) + e//f = ((ad + bc)//(bd)) + e//f \\ &= (adf + bcf + bde)//(bde).\end{aligned}$$

$$\begin{aligned}x + (y + z) &= (a//b) + ((c//d) + (e//f)) = (a//b) + ((cd + de)//(df)) \\ &= (adf + bcf + bde)//(bde).\end{aligned}$$

So, $(x + y) + z = x + (y + z)$, as desired. □

Definition 4.9 (Quotient). Let x, y be rational numbers such that $y \neq 0$. We define the **quotient** x/y of x and y by

$$x/y := x \times y^{-1}.$$

Remark 4.10. For any integers a, b with $b \neq 0$, note that $a/b = a//b$, since

$$a/b = ab^{-1} = (a//1) \times (1//b) = a//b.$$

So, from now on, we use the notation a/b instead of $a//b$.

Remark 4.11. From now on, we will use the field axioms of Proposition 4.8 without explicit reference.

As in the case of integers, we now define positive and negative rational numbers.

Definition 4.12. A rational number x is said to be **positive** if and only if $x = a/b$ for some positive integers a, b. A rational number x is said to be **negative** if and only if $x = -y$ for a positive rational number y.

Remark 4.13. A positive integer is a positive rational number, and a negative integer is a negative rational number, so our notions of positive and negative are consistent.

Lemma 4.14. *Let x be a rational number. Then exactly one of the following three statements is true.*

- *x is equal to 0.*
- *x is a positive rational number.*
- *x is a negative rational number.*

We now define an order on the rationals that extends the notion of order on the integers.

Definition 4.15 (**Order**)**.** Let x, y be rational numbers. We write $x > y$ if and only if $x - y$ is a positive rational number. We write $x < y$ if and only if $y - x$ is a positive rational number. We write $x \geq y$ if and only if either $x > y$ or $x = y$. We write $x \leq y$ if and only if either $x < y$ or $x = y$.

Proposition 4.16 (**Properties of Order**)**.** *Let x, y, z be rational numbers. Then*

(1) *Exactly one of the statements $x = y$, $x < y$, $x > y$ is true.*
(2) *$x < y$ if and only if $y > x$.*
(3) *If $x < y$ and $y < z$, then $x < z$*
(4) *If $x < y$, then $x + z < y + z$.*
(5) *If $x < y$ and if z is positive, then $xz < yz$.*

Remark 4.17. The five properties of Proposition 4.16 combined with the field axioms of Proposition 4.8 say that the set of rational numbers $\mathbb{Q}$ form an **ordered field**.

Unlike the integers, the rationals have the following density property. Given any two rational numbers, there is a third rational number between them.

Proposition 4.18. *Given any two rational numbers x, z with $x < z$, there exists a rational number y such that $x < y < z$.*

Proof. Define $y := (x + z)/2$. Since $x < z$ and $1/2$ is positive, Proposition 4.16(5) says that $x/2 < z/2$. Adding $z/2$ to both sides and using Proposition 4.16(4), we get $x/2 + z/2 < z/2 + z/2 = z$. That is, $y < z$. Adding $x/2$ to both sides of $x/2 < z/2$, we get $x = x/2 + x/2 < x/2 + z/2$. That is, $x < y$. In conclusion, $x < y < z$, as desired. □

Even though the rationals have some density in the sense of Proposition 4.18, the set of rational numbers still has many gaps. To illustrate this fact, consider the following classical proposition.

Proposition 4.19. *There does not exist a rational number x such that $xx = 2$.*

Proof. We argue by contradiction. Assume that x is rational and $xx = 2$. We may assume that x is positive, since $xx = (-x)(-x)$. Let p, q be integers with $q \neq 0$ such that $x = p/q$. Since x is positive, we may assume that p, q are natural numbers. Since $xx = 2$, we have $pp = 2qq$. Recall that a natural number a is **even** if there exists a natural number b such that $a = 2b$, and a natural number a is **odd** if there exists a natural number b such that $a = 2b+1$. Note that every natural number is either even or odd, and natural number cannot be both even and odd. Both of these facts follow from Proposition 2.34. If a is odd, note that $aa = 4bb+2b+2b+1 = 2(2bb+b+b)+1$, so aa is odd. So, by taking the contrapositive: if aa is even, then a is even. Since $pp = 2qq$, pp is even, so we conclude that p is even, so there exists a natural number k such that $p = 2k$. Since p is positive, k is positive. Since

$pp = 2qq$, we get $pp = 4kk = 2qq$, so $qq = 2kk$. Since $pp = 2qq$, and p, q are positive, we have $q < p$.

In summary, we started with positive natural numbers p, q such that $pp = 2qq$. And we now have positive natural numbers q, k such that $qq = 2kk$, and such that $q < p$. We can therefore iterate this procedure. For any natural number n, suppose inductively we have p_n, q_n positive natural numbers such that $p_n p_n = 2q_n q_n$. Then we have found natural numbers p_{n+1}, q_{n+1} such that $p_{n+1}p_{n+1} = 2q_{n+1}q_{n+1}$, and such that $p_{n+1} < p_n$. The existence of the natural numbers $p_1, p_2, \ldots$ violates the principle of infinite descent (Exercise 4.20), so we have obtained a contradiction. We conclude that no rational x satisfies $xx = 2$. □

Exercise 4.20. Prove the principle of infinite descent. Let $p_0, p_1, p_2, \ldots$ be an infinite sequence of natural numbers such that $p_0 > p_1 > p_2 > \cdots$. Prove that no such sequence exists. (Hint: Assume by contradiction that such a sequence exists. Then prove by induction that for all natural numbers n, N, we have $p_n \geq N$. Use this fact to obtain a contradiction.)

4.1. **Operations on Rationals.** We now introduce a few additional operations on the rationals $\mathbb{Q}$. These operations will help in our construction of the real numbers.

Definition 4.21 (**Absolute Value**). Let x be a rational number. The **absolute value** $|x|$ of x is defined as follows. If $x \geq 0$, then $|x| := x$. If $x < 0$, then $|x| := -x$.

Definition 4.22 (**Distance**). Let x, y be rational numbers. The quantity $|x - y|$ is called the **distance between** x and y. We denote $d(x, y) := |x - y|$.

The following inequalities will be used very often in this course.

Proposition 4.23. *Let x, y be rational numbers. Then $|x| \geq 0$, and $|x| = 0$ if and only if $x = 0$. We also have the* ***triangle inequality***

$$|x + y| \leq |x| + |y|,$$

the bounds

$$-|x| \leq x \leq |x|$$

and the equality

$$|xy| = |x|\,|y|.$$

In particular,

$$|-x| = |x|.$$

Also, the distance $d(x, y)$ satisfies the following properties. Let x, y, z be rational numbers. Then $d(x, y) = 0$ if and only if $x = y$. Also, $d(x, y) = d(y, x)$. Lastly, we have the triangle inequality

$$d(x, z) \leq d(x, y) + d(y, z).$$

Exercise 4.24. By breaking into different cases as necessary, prove Proposition 4.23.

Exercise 4.25. Using the usual triangle inequality, prove the **reverse triangle inequality**: For any rational numbers x, y, we have $|x - y| \geq ||x| - |y||$.

Definition 4.26 (**Exponentiation**). Let x be a rational number. We define $x^0 := 1$. Now, let n be any natural number, and suppose we have inductively defined x^n. Then define $x^{n+1} := x^n \times x$.

The following properties of exponentiation then follow by induction.

Proposition 4.27. *Let x, y be rational numbers, and let n, m be natural numbers.*

- $x^n x^m = x^{n+m}$, $(x^n)^m = x^{nm}$, *and* $(xy)^n = x^n y^n$.
- $x^n = 0$ *if and only if* $x = 0$ *and* $n > 0$.
- *If* $x \geq y \geq 0$, *then* $x^n \geq y^n \geq 0$.
- $|x|^n = |x^n|$.

Definition 4.28 (Negative Exponentiation). Let x be a nonzero rational number, and let n be a positive natural number. Define $x^{-n} := 1/x^n$.

Proposition 4.29. *Let x, y be nonzero rational numbers, and let n, m be integers.*

- $x^n x^m = x^{n+m}$, $(x^n)^m = x^{nm}$, *and* $(xy)^n = x^n y^n$.
- *If* $x \geq y > 0$, *then* $x^n \geq y^n > 0$ *if* $n > 0$, *and* $0 < x^n \leq y^n$ *if* $n < 0$.
- $|x|^n = |x^n|$.

5. Cauchy Sequences of Rationals

Having established many properties of the rational numbers, we can finally begin to construct the real number system. As we saw in Proposition 4.19, there does not exist a rational number x such that $x^2 = 2$. Nevertheless, we can still find rational numbers x such that x^2 becomes as close as desired to 2. In this sense, the rational numbers have gaps between them. And filling in these gaps will exactly give us the real number system. There are a few different ways to fill in these gaps between the rational numbers. We will discuss the method of Cauchy sequences, since their investigation will lead naturally to further topics of interest.

As a preliminary result, we consider the gaps between the integers.

Proposition 5.1. *Let x be a rational number. Then there exists a unique integer n such that $n \leq x < n + 1$. In particular, there exists an integer N such that $x < N$.*

Exercise 5.2. Using the Euclidean Algorithm (Proposition 2.34), prove Proposition 5.1.

Proposition 5.3. *For any rational number $\varepsilon > 0$, there exists a nonnegative rational number x such that $x^2 < 2 < (x + \varepsilon)^2$.*

Proof. We argue by contradiction. Suppose there exists $\varepsilon > 0$ and there does not exist a nonnegative rational number x such that $x^2 < 2 < (x + \varepsilon)^2$. So, every nonnegative rational number x with $x^2 < 2$ must also satisfy $(x + \varepsilon)^2 \leq 2$. From Proposition 4.19, $(x + \varepsilon)^2 \neq 2$, so $(x + \varepsilon)^2 < 2$. Note that $(x + \varepsilon)^2$ is rational and $(x + \varepsilon)^2 < 2$, so using this number in place of x, we see that we must have $(x + 2\varepsilon)^2 < 2$ as well. Indeed, an inductive argument shows that, for any natural number n, $(x + n\varepsilon)^2 < 2$. Choosing $x = 0$, we see that $(n\varepsilon)^2 < 2$, for any natural number n. However, since $2/\varepsilon$ is rational, Proposition 5.1 says that there exists an integer N such that $N > 2/\varepsilon$. That is, $N\varepsilon > 2$, so $(N\varepsilon)^2 > 4$. This inequality contradicts that $(N\varepsilon)^2 < 2$. Since we have arrived at a contradiction, we conclude that an x exists satisfying the proposition. □

Indeed, we "know" that the sequence of rational numbers

$$1.4, \quad 1.41, \quad 1.414, \quad 1.4142, \quad \ldots$$

becomes arbitrarily close to a number x such that $x^2 = 2$. And this sort of sequential procedure is exactly how we will construct the rational numbers. Note that we define the decimal 1.4142 as the rational number 14142/10000.

Definition 5.4 (**Sequence of rationals**). Let m be an integer. A **sequence** $(a_n)_{n=m}^{\infty}$ **of rationals** is any function from the set $\{n \in \mathbb{N} \colon n \geq m\}$ to $\mathbb{Q}$. Informally, a sequence of rationals is an ordered list of rational numbers.

Example 5.5. The sequence $(n^2)_{n=0}^{\infty}$ is the collection $0, 1, 4, 9, 16, \ldots$ of natural numbers.

We will define real numbers as certain limits of sequences of rationals. A general sequence of rationals does not seem to have a sensible limit, so we need to restrict the sequences that we are considering. For example, the sequence $((-1)^n)_{n=0}^{\infty}$ does not seem to have any sensible limit. The following definition states precisely what kind of sequences we would like to focus on. The idea is that, eventually, the sequence elements need to be close to each other. This vague statement is then formalized as follows.

Definition 5.6 (**Cauchy sequence**). A sequence $(a_n)_{n=0}^{\infty}$ of rational numbers is said to be a **Cauchy sequence** if and only if, for every rational $\varepsilon > 0$, there exists a natural number $N = N(\varepsilon)$ such that, for all $j, k \geq N$, we have $d(a_j, a_k) < \varepsilon$.

Example 5.7. The sequence $(1/n)_{n=1}^{\infty}$ is a Cauchy sequence. To see this, let $\varepsilon > 0$ be a rational number. From Proposition 5.1, let N be a natural number such that $N > 2/\varepsilon$. Then $1/N < \varepsilon/2$. Now, let $j, k \geq N$ so that $1/j \leq 1/N$ and $1/k \leq 1/N$. From the triangle inequality, we then have

$$d(1/j, 1/k) = |1/j - 1/k| \leq |1/j| + |1/k| = 1/j + 1/k \leq 2/N < \varepsilon.$$

To get an idea of where we are headed, we are going to *define* the real numbers to be the "limits" of Cauchy sequences. In order to make this statement rigorous, we need to show that a Cauchy sequence has a limit, and we need to discuss when two Cauchy sequences have the same limit. If two Cauchy sequences have the same limit, we will say that they are equal. Before defining the real numbers, we need some preliminary facts about Cauchy sequences.

Definition 5.8 (**Bounded Sequence**). Let $M \geq 0$ be rational. A finite sequence of rationals $a_0, \ldots, a_n$ is **bounded by** M if and only if $|a_i| \leq M$ for all $i \in \{0, \ldots, n\}$. An infinite sequence of rationals $(a_i)_{i=0}^{\infty}$ is **bounded by** M if and only if $|a_i| \leq M$ for all $i \in \mathbb{N}$. A sequence $(a_i)_{i=0}^{\infty}$ is **bounded** if and only if there exists a positive rational M such that $(a_i)_{i=0}^{\infty}$ is bounded by M.

Lemma 5.9. *Every Cauchy sequence is bounded.*

Exercise 5.10. Prove Lemma 5.9

Definition 5.11 (**Equivalent Cauchy Sequences**). Let $(a_n)_{n=0}^{\infty}, (b_n)_{n=0}^{\infty}$ be Cauchy sequences. We say that these Cauchy sequences are **equivalent** if and only if, for every rational $\varepsilon > 0$, there exists a natural number $N = N(\varepsilon) \geq 0$ such that $|a_n - b_n| < \varepsilon$ for all $n \geq N$.

As with our notations of equivalence of integers and rationals, we need to show that this notion of equivalence is an equivalence relation. That is, we need the following three properties.

Lemma 5.12. *Let* $(a_n)_{n=0}^{\infty}, (b_n)_{n=0}^{\infty}, (c_n)_{n=0}^{\infty}$ *be Cauchy sequences.*

- $(a_n)_{n=0}^{\infty}$ *is equivalent to* $(a_n)_{n=0}^{\infty}$.

- *If* $(a_n)_{n=0}^{\infty}$ *is equivalent to* $(b_n)_{n=0}^{\infty}$*, then* $(b_n)_{n=0}^{\infty}$ *is equivalent to* $(a_n)_{n=0}^{\infty}$*.*
- *If* $(a_n)_{n=0}^{\infty}$ *is equivalent to* $(b_n)_{n=0}^{\infty}$*, and if* $(b_n)_{n=0}^{\infty}$ *is equivalent to* $(c_n)_{n=0}^{\infty}$*, then* $(a_n)_{n=0}^{\infty}$ *is equivalent to* $(c_n)_{n=0}^{\infty}$*.*

Proof. We prove the third item. Let $\varepsilon > 0$ be a rational number. Note that $\varepsilon/2 > 0$ is a rational number. So, by assumption, there exist $L, M > 0$ such that, for all $n \geq L$, $|a_n - b_n| < \varepsilon/2$, and for all $n \geq M$, $|b_n - c_n| < \varepsilon/2$. Define $N := \max(L, M)$. Then, for all $n \geq N$, we have by the triangle inequality

$$|a_n - c_n| = |a_n - b_n + b_n - c_n| \leq |a_n - b_n| + |b_n - c_n| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

That is, $(a_n)_{n=0}^{\infty}$ is equivalent to $(c_n)_{n=0}^{\infty}$, as desired. □

Remark 5.13. The above proof strategy occurs very often in analysis, so it should be ingrained in your memory. The idea is that, in order to prove that two things are close, you add and subtract the same number, and then apply the triangle inequality.

6. Construction of the Real Numbers

We can now finally give a definition of a real number. As in our construction of the integers and rational numbers, we will begin by using some artificial symbol to designate a real number. However, the construction of the real numbers requires a new ingredient, which is the Cauchy sequence of rational numbers.

Definition 6.1 (**Real Number**)**.** A **real number** is an object of the form $\mathrm{LIM}_{n\to\infty} a_n$, where $(a_n)_{n=0}^{\infty}$ is a Cauchy sequence. Two real numbers $\mathrm{LIM}_{n\to\infty} a_n$, $\mathrm{LIM}_{n\to\infty} b_n$ are equal if and only if $(a_n)_{n=0}^{\infty}, (b_n)_{n=0}^{\infty}$ are equivalent Cauchy sequences. The set of all real numbers is denoted by $\mathbb{R}$

Remark 6.2. We refer to $\mathrm{LIM}_{n\to\infty} a_n$ as the **formal limit** of the Cauchy sequence $(a_n)_{n=0}^{\infty}$. Later on, we will show that a Cauchy sequence has an actual limit as $n \to \infty$, which explains our use of this notation.

Even though we define real numbers in terms of Cauchy sequences, which allows us to axiomatize the real number system and prove facts about this system, our approach perhaps does not have many direct consequences for other results concerning real numbers and functions. To use an analogy, even though we know that all materials in the world are made of atoms, this fact only marginally affects our material interaction with the physical world. On the other hand, the exact way that we construct and analyze the real numbers *does* influence our understanding of other mathematical objects. To use the same analogy as before, our understanding of atoms *does* allow us to better understand some things that we encounter in the physical world, such as light, the sun, etc.

As in our treatment of the integers and rationals, we now define arithmetic on the real numbers.

Definition 6.3 (**Addition of Real Numbers**)**.** Let $x = \mathrm{LIM}_{n\to\infty} a_n$ and let $y = \mathrm{LIM}_{n\to\infty} b_n$ be real numbers. Then define the sum of x and y by $x + y := \mathrm{LIM}_{n\to\infty}(a_n + b_n)$.

We now check that addition of two real numbers give a real number, and that addition is well-defined.

Lemma 6.4. *Let $x = \mathrm{LIM}_{n\to\infty} a_n$ and let $y = \mathrm{LIM}_{n\to\infty} b_n$ be real numbers. Then $x + y$ is also a real number.*

Proof. We need to show that $(a_n + b_n)_{n=0}^{\infty}$ is a Cauchy sequence. The proof is similar to that of Lemma 5.12. Let $\varepsilon > 0$ be a rational number. Note that $\varepsilon/2 > 0$ is a rational number. By assumption, there exist $L, M > 0$ such that, for all $j, k \geq L$, $|a_j - a_k| < \varepsilon/2$, and for all $j, k \geq M$, $|b_j - b_k| < \varepsilon/2$. Define $N := \max(L, M)$. Then, for all $j, k \geq N$, we have by the triangle inequality

$$|a_j + b_j - a_k - b_k| = |a_j - a_k + b_j - b_k| \leq |a_j - a_k| + |b_j - b_k| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

That is, $(a_n + b_n)_{n=0}^{\infty}$ is a Cauchy sequence, as desired. □

Lemma 6.5. *Let $x = \mathrm{LIM}_{n\to\infty} a_n$ and let $y = \mathrm{LIM}_{n\to\infty} b_n$ be real numbers. Let $x' = \mathrm{LIM}_{n\to\infty} a'_n$ be a real number such that $x = x'$. Then $x + y = x' + y$.*

Proof. Let $\varepsilon > 0$ be a rational number. Since $x = x'$, there exists $N > 0$ such that, for all $n \geq N$, $|a_n - a'_n| < \varepsilon$. Then, for all $n \geq N$,

$$|a_n + b_n - a'_n - b_n| = |a_n - a'_n| < \varepsilon.$$

That is, $(a_n + b_n)_{n=0}^{\infty}$ is equivalent to $(a'_n + b_n)_{n=0}^{\infty}$, as desired. □

Remark 6.6. If additionally y' is equivalent to y, then $x + y = x + y'$. To see this, note that addition is commutative for real numbers, which follows from the commutativity of addition for rational numbers.

We now define multiplication.

Definition 6.7 (Multiplication of Real Numbers). Let $x = \mathrm{LIM}_{n\to\infty} a_n$ and let $y = \mathrm{LIM}_{n\to\infty} b_n$ be real numbers. Define the product $xy := \mathrm{LIM}_{n\to\infty}(a_n b_n)$.

Proposition 6.8. *Let $x = \mathrm{LIM}_{n\to\infty} a_n$ and let $y = \mathrm{LIM}_{n\to\infty} b_n$ be real numbers. Then xy is a real number. Also if $x' = \mathrm{LIM}_{n\to\infty} a'_n$ is a real number such that $x = x'$, then $xy = x'y$.*

Exercise 6.9. Prove Proposition 6.8.

Remark 6.10. We can now realize the rational numbers as a subset of the real numbers. Given a rational number $q \in \mathbb{Q}$, consider the constant Cauchy sequence $q, q, q, q, \ldots$. Then addition and multiplication are identical for $q \in \mathbb{Q}$ and for the Cauchy sequence $q, q, q, q, \ldots$. Moreover, this identification of rational numbers within the real numbers is consistent with our two notions of equality. That is, $p, q \in \mathbb{Q}$ are equal if and only if the Cauchy sequences $p, p, p, \ldots$ and $q, q, q, \ldots$ are equal.

Definition 6.11. Since we have defined multiplication of real numbers, we can now define the **negation** of a real number x by

$$-x := (-1) \times x.$$

We therefore see that

$$-(\mathrm{LIM}_{n\to\infty} a_n) = \mathrm{LIM}_{n\to\infty}(-a_n).$$

Also, we define **subtraction** of real numbers x, y by

$$x - y := x + (-y).$$

We therefore see that

$$\mathrm{LIM}_{n\to\infty}a_n - (\mathrm{LIM}_{n\to\infty}b_n) = \mathrm{LIM}_{n\to\infty}(a_n - b_n).$$

We will now show that the real number system satisfies all of the usual algebraic identities with which we are acquainted. That is, the number system $\mathbb{R}$ is a **field**. The final property of the field, the multiplicative inverse, is a bit tricky to verify, so we will deal with that last. That is, we will first only assert that $\mathbb{R}$ is a commutative ring.

Proposition 6.12. *Let x, y, z be real numbers. Then the following laws of algebra hold.*

- $x + y = y + x$ *(Commutativity of addition)*
- $(x + y) + z = x + (y + z)$ *(Associativity of addition)*
- $x + 0 = 0 + x = x$ *(Additive identity element)*
- $x + (-x) = (-x) + x = 0$ *(Additive inverse)*
- $xy = yx$ *(Commutativity of multiplication)*
- $(xy)z = x(yz)$ *(Associativity of multiplication)*
- $x1 = 1x = x$ *(Multiplicative identity element)*
- $x(y + z) = xy + xz$ *(Left Distributivity)*
- $(y + z)x = yx + zx$ *(Right Distributivity)*

Proof. We only prove the associativity of a multiplication, the others being similar. As we will see, these properties follow readily from the corresponding properties of the rational numbers. Let x, y, z be real numbers. Write $x = \mathrm{LIM}_{n\to\infty}a_n$, $y = \mathrm{LIM}_{n\to\infty}b_n$, $z = \mathrm{LIM}_{n\to\infty}c_n$. Then $(xy) = \mathrm{LIM}_{n\to\infty}(a_nb_n)$, and $(xy)z = \mathrm{LIM}_{n\to\infty}[(a_nb_n)c_n]$. From associativity of multiplication of rationals, we then have

$$(xy)z = \mathrm{LIM}_{n\to\infty}[a_n(b_nc_n)] = x \times \mathrm{LIM}_{n\to\infty}(b_nc_n) = x(yz),$$

as desired. □

We now need to define the reciprocal. Note that we cannot simply define the reciprocal of a Cauchy sequence $a_0, a_1, \ldots$ to be the sequence a_0^{-1}, a_1^{-1}, since some of the elements of the sequence $a_0, a_1, \ldots$ could be zero. Thankfully, this problem can be circumvented by simply waiting for the Cauchy sequence to be nonzero.

Lemma 6.13. *Let x be a nonzero real number. Then there exists a rational number $\varepsilon > 0$ such that, for any Cauchy sequence $(a_n)_{n=0}^{\infty}$ with $x = \mathrm{LIM}_{n\to\infty}a_n$, there exists $N > 0$ such that, for all $n \geq N$, $|a_n| > \varepsilon$. In this statement, note that ε does not depend on the Cauchy sequence, but N does.*

Proof. Since x is nonzero, $(a_n)_{n=0}^{\infty}$ is not equivalent to the Cauchy sequence $0, 0, 0, \ldots$. So, negating the statement "$(a_n)_{n=0}^{\infty}$ is equivalent to $0, 0, 0, \ldots,$ " we get the following. There exists a rational $\varepsilon > 0$ such that, for all natural numbers $L > 0$, there exists $\ell > L$ such that $|a_\ell| \geq 3\varepsilon$. Since $(a_n)_{n=0}^{\infty}$ is a Cauchy sequence, there exists $M > 0$ such that, for all $j, k > M$, we have $|a_j - a_k| < \varepsilon$. So, if we choose $L := M$, there exists $\ell > L = M$ such that $|a_\ell| \geq 3\varepsilon$. So, for any $n > \ell > M$, we have by Exercise 4.25

$$|a_n| = |a_n - a_\ell + a_\ell| \geq |a_\ell| - |a_n - a_\ell| > 3\varepsilon - \varepsilon = 2\varepsilon.$$

So, the assertion is proven with an ε that may depend on the chosen Cauchy sequence $(a_n)_{n=0}^{\infty}$. To see that we can choose ε to not depend on the particular Cauchy sequence, let $(a'_n)_{n=0}^{\infty}$ be any Cauchy sequence equivalent to $(a_n)_{n=0}^{\infty}$. That is, there exists $K > 0$ such that, for all $n > K$, we have $|a_n - a'_n| < \varepsilon$. Finally, define $N := \max(\ell, K)$. Then, for any $n > N$, we have

$$|a'_n| = |a'_n - a_n + a_n| \geq |a_n| - |a_n - a'_n| \geq 2\varepsilon - \varepsilon = \varepsilon.$$

Since $(a'_n)_{n=0}^{\infty}$ is any Cauchy sequence equivalent to $(a_n)_{n=0}^{\infty}$, we have shown that the number ε does not depend on the particular Cauchy sequence, as desired. □

With this lemma, we can now define the inverse of a real number.

Definition 6.14 (Inverse). Let x be a nonzero real number. Let $(a_n)_{n=0}^{\infty}$ be any Cauchy sequence with $x = \mathrm{LIM}_{n\to\infty} a_n$. From Lemma 6.13, there exists a rational $\varepsilon > 0$ and a natural number $N > 0$ such that, for all $n > N$, $|a_n| > \varepsilon > 0$. Consider the equivalent Cauchy sequence b_n where $b_n := a_n$ for all $n > N$, and $b_n := 1$ for all $0 \leq n \leq N$. Then $x = \mathrm{LIM}_{n\to\infty} b_n$, and $|b_n| > \varepsilon$ for all $n \geq 0$. So, we define the **reciprocal** x^{-1} of x as $x^{-1} := \mathrm{LIM}_{n\to\infty}(b_n^{-1})$.

We now need to check that x^{-1} is a real number, and also that x^{-1} is well-defined. That is, we need to show that x^{-1} does not depend on the Cauchy sequence $(a_n)_{n=0}^{\infty}$.

Lemma 6.15. *Let $\delta > 0$. Let $(a_n)_{n=0}^{\infty}$ be a Cauchy sequence such that $|a_n| > \delta$ for all $n \geq 0$. Then $(a_n^{-1})_{n=0}^{\infty}$ is a Cauchy sequence.*

Proof. Let $\varepsilon > 0$. Since $|a_n| > \delta > 0$ for all $n \geq 0$, we have $|a_n|^{-1} < 1/\delta$ for all $n \geq 0$. Since $(a_n)_{n=0}^{\infty}$ is a Cauchy sequence, there exists $N > 0$ such that, for all $j, k > N$, we have $|a_j - a_k| < \varepsilon\delta^2$. Then, for all $j, k > N$, we have

$$\left|a_j^{-1} - a_k^{-1}\right| = |a_j|^{-1} |a_k|^{-1} |a_k - a_j| < \delta^{-2}\varepsilon\delta^2 = \varepsilon.$$

That is, the sequence $(a_n^{-1})_{n=0}^{\infty}$ is a Cauchy sequence. □

Lemma 6.16. *Let x be a nonzero real number. Let $(a_n)_{n=0}^{\infty}$ and $(a'_n)_{n=0}^{\infty}$ be Cauchy sequences such that $x = \mathrm{LIM}_{n\to\infty} a_n$ and such that $x = \mathrm{LIM}_{n\to\infty} a'_n$. Then, after changing a finite number of terms of these Cauchy sequences, we have: $\mathrm{LIM}_{n\to\infty} a_n^{-1}$ is equivalent to $\mathrm{LIM}_{n\to\infty}(a'_n)^{-1}$.*

Proof. Let $\varepsilon > 0$. From Lemma 6.13, let $\delta > 0$ and let $L > 0$ such that, for all $n > L$, $|a_n| > \delta$ and $|a'_n| > \delta$. Since $(a_n)_{n=0}^{\infty}$ and $(a'_n)_{n=0}^{\infty}$ are equivalent, there exists $M > 0$ such that, for all $n > M$, we have $|a_n - a'_n| < \varepsilon\delta^2$. Define $N := \max(L, M)$. Then, for all $n > N$,

$$\left|a_n^{-1} - (a'_n)^{-1}\right| = |a_n|^{-1} |a'_n|^{-1} |a_n - a'_n| < \delta^{-2}\varepsilon\delta^2 = \varepsilon.$$

So, if we define $b_n := a_n$ for all $n > N$, $b'_n := a'_n$ for all $n \geq N$, and $b_n = b'_n = 1$ for all $0 \leq n \leq N$, we see that $\mathrm{LIM}_{n\to\infty} b_n^{-1}$ is equivalent to $\mathrm{LIM}_{n\to\infty}(b'_n)^{-1}$, as desired. □

Lemma 6.15 shows that x^{-1} is a real number whenever x is a nonzero real number. And Lemma 6.16 shows that x^{-1} is well-defined.

Remark 6.17. If x is a nonzero real number, it follows from Definition 6.14 that $xx^{-1} = x^{-1}x = 1$. Combining this fact with Proposition 6.12, we conclude that $\mathbb{R}$ is a field, as previously asserted.

Remark 6.18. Note that our definition of reciprocal is consistent with the definition of reciprocal of a rational number.

Definition 6.19 (**Division**)**.** Let x, y be real numbers with y nonzero. We then define $x/y := x \times y^{-1}$. We then have the **cancellation law** (which follows from the same property for rational numbers). If x, y, z are real numbers with z nonzero, and if $xz = yz$, then $x = y$.

Remark 6.20. We now have all of the usual arithmetic operations on the real numbers. We now turn to the order properties of the reals. Note that we cannot simply say that: a Cauchy sequence is positive if and only if its elements are all positive. For example, the Cauchy sequence $-1, 1, 1, 1, 1, 1, 1, 1, 1, \ldots$ corresponds to the positive real number 1, but it has a negative value in the sequence. For another example, note that the Cauchy sequence $1, 1/2, 1/3, 1/4, 1/5, \ldots$ has all positive elements, but it is equivalent to the sequence $0, 0, 0 \ldots$, which is certainly not positive. So, we need to be careful in defining positivity.

6.1. **Ordering of the Reals.**

Definition 6.21. A real number x is said to be **positive** if and only if there exists a positive rational $\varepsilon > 0$ such that, for any Cauchy sequence $(a_n)_{n=0}^{\infty}$ with $x = \mathrm{LIM}_{n\to\infty}(a_n)$, there exists a natural number $N > 0$ such that, for all $n > N$, we have $a_n > \varepsilon > 0$. A real number x is said to be **negative** if and only if $-x$ is positive.

Remark 6.22. Note that these definitions are consistent with the definitions of positivity and negativity for rational numbers. For example, if $x > 0$ is rational, then Lemma 6.13 implies that there exists $\varepsilon > 0$ such that, for any Cauchy sequence $(a_n)_{n=0}^{\infty}$ with $x = \mathrm{LIM}_{n\to\infty} a_n$ there exists $N > 0$ such that for all $n > N$, $a_n > \varepsilon > 0$. (You will investigate the details of this argument in Exercise 6.31.)

Proposition 6.23. *For every real number x, exactly one of the following statements is true: x is positive, x is negative, or x is zero. If x, y are positive real numbers, then $x + y$ is positive, and xy is positive.*

Exercise 6.24. Using Lemma 6.13, prove Proposition 6.23

We can now define order, since we have just defined positivity and negativity.

Definition 6.25. Let x, y be real numbers. We say that x **is greater than** y, and we write $x > y$ if and only if $x - y$ is a positive real number. We say that x is **less than** y, and we write $x < y$ if and only if $y - x$ is a positive real number. We write $x \geq y$ if and only if $x > y$ or $x = y$, and we similarly define $x \leq y$.

Remark 6.26. This ordering on the reals is consistent with the ordering we gave for the rational numbers. That is, if a, b are two rational numbers with $a < b$, then the real numbers a, b also satisfy $a < b$. And similarly for the assertion $a > b$.

The real numbers now satisfy all of the same axioms for order than the rational numbers satisfied in Proposition 4.16.

Proposition 6.27 (**Properties of Order**)**.** *Let x, y, z be real numbers. Then*

(1) *Exactly one of the statements $x = y$, $x < y$, $x > y$ is true.*
(2) *$x < y$ if and only if $y > x$.*
(3) *If $x < y$ and $y < z$, then $x < z$*
(4) *If $x < y$, then $x + z < y + z$.*
(5) *If $x < y$ and if z is positive, then $xz < yz$.*

Remark 6.28. In conclusion, the real numbers form an **ordered field**.

Proof. We only prove (5), since the other proofs similarly follow from Proposition 6.23 and basic algebra. Suppose $x < y$ and z is positive. Since $x < y$, $y - x$ is positive. So, from Proposition 6.23, $z(y - x)$ is positive, so $xz < yz$, as desired. □

Proposition 6.29. *Let x be a positive real number. Then x^{-1} is also a positive real number. If y is a positive real number with $x > y$, then $x^{-1} < y^{-1}$.*

Proof. Let x be a positive real number. Since $xx^{-1} = 1$, the real number x^{-1} is nonzero. (If we had $x^{-1} = 0$, then $xx^{-1} = 0$.) We show that x^{-1} is positive by contradiction. If x^{-1} were not positive, it would be negative, since $x^{-1} \neq 0$. From Proposition 6.23, we get that xx^{-1} is negative, contradicting that $xx^{-1} = 1$. We therefore conclude that x^{-1} is positive.

We now show that $x^{-1} < y^{-1}$ by contradiction. Assume that $x^{-1} \geq y^{-1}$. Then from Proposition 6.27(5) applied twice, $xx^{-1} \geq xy^{-1} > yy^{-1}$, i.e. $1 > 1$, a contradiction. We conclude that $x^{-1} < y^{-1}$, as desired. □

Proposition 6.30. *Let x, y be real numbers. Suppose $(a_n)_{n=0}^{\infty}, (b_n)_{n=0}^{\infty}$ are Cauchy sequences with $x = \mathrm{LIM}_{n\to\infty} a_n$ and $y = \mathrm{LIM}_{n\to\infty} b_n$. Assume that there exists $N > 0$ such that for all $n > N$, we have $a_n \leq b_n$. Then $x \leq y$.*

Proof. We argue by contradiction. Suppose $x > y$. Then $x - y$ is positive. Note that $(a_n - b_n)_{n=0}^{\infty}$ is a Cauchy sequence such that $x - y = \mathrm{LIM}_{n\to\infty}(a_n - b_n)$. So, by Definition 6.21, there exists $\delta > 0$ and there exists $M > 0$ such that, for all $n > M$, we have $a_n - b_n > \delta > 0$. In particular, we have $a_{M+1} > b_{M+1}$, a contradiction. Since we have achieved a contradiction, we are done. □

Exercise 6.31. Prove the following variant of Lemma 6.13: Let x be a positive real number. Then there exists a rational number $\varepsilon > 0$ such that, for any Cauchy sequence $(a_n)_{n=0}^{\infty}$ with $x = \mathrm{LIM}_{n\to\infty} a_n$, there exists $N > 0$ such that, for all $n \geq N$, $a_n > \varepsilon$. In this statement, note that ε does not depend on the Cauchy sequence, but N does. (And similarly, when x is a negative real number.)

Remark 6.32. Since we have defined positive and negative real numbers, we can then define the absolute value $|x|$ exactly as in Definition 4.21. We then define $d(x, y) := |x - y|$ just as before, but now for real numbers x, y. Note that, if $(a_n)_{n=0}^{\infty}$ is a Cauchy sequence such that $x = \mathrm{LIM}_{n\to\infty} a_n$, then $|a_n|$ is a Cauchy sequence for $|x|$, by Exercise 6.31.

Theorem 6.33 (**Triangle Inequality for Real Numbers**)**.** *Let x, y be real numbers. Then $|x + y| \leq |x| + |y|$.*

Proof. Suppose $(a_n)_{n=0}^{\infty}, (b_n)_{n=0}^{\infty}$ are Cauchy sequences with $x = \mathrm{LIM}_{n\to\infty} a_n$, $y = \mathrm{LIM}_{n\to\infty} b_n$. From the triangle inequality for rational numbers (Proposition 4.23), $|a_n + b_n| \leq |a_n| + |b_n|$ for all $n \in \mathbb{N}$. By Remark 6.32, note that $(|a_n|)_{n=0}^{\infty}$ is a Cauchy sequence for $|x|$, and $(|b_n|)_{n=0}^{\infty}$ is a Cauchy sequence for $|y|$, and $(|a_n + b_n|)_{n=0}^{\infty}$ is a Cauchy sequence for $|x + y|$. Since $|a_n + b_n| \leq |a_n| + |b_n|$ for all $n \in \mathbb{N}$, Proposition 6.30 implies $|x + y| \leq |x| + |y|$. □

Theorem 6.34 (**The Rationals are Dense in the Real Numbers**)**.** *Let x be a real number and let $\varepsilon > 0$ be any rational number. Then there exists a rational number y such that $|x - y| < \varepsilon$.*

Exercise 6.35. Prove Theorem 6.34.

Theorem 6.36 (**Archimedean Property**). *Let x, ε be any positive real numbers. Then there exists a positive integer N such that $N\varepsilon > x$.*

Proof. From Propositions 6.29 and 6.23, ε/x is a positive real number. Let $(a_n)_{n=0}^{\infty}$ be a Cauchy sequence of rationals such that $\varepsilon/x = \mathrm{LIM}_{n\to\infty}a_n$. From Exercise 6.31, there exists a rational number y and there exists a natural number M such that, for all $n > M$, we have $a_n > y > 0$. Write $y = p/q$ with $p, q \in \mathbb{N}$, $p \neq 0$, $q \neq 0$. Then $a_n > y \geq 1/q > 0$, so $\varepsilon/x \geq 1/q$ by Proposition 6.30, so $(q+1)\varepsilon > x$. Setting $N := q+1$ completes the proof. □

Corollary 6.37. *Let x, z be real numbers with $x < z$. Then there exists a rational number y with $x < y < z$.*

Exercise 6.38. Using Theorems 6.34 and 6.36, prove Corollary 6.37.

7. The Least Upper Bound Property

We have constructed the real numbers, defined their arithmetic operations, and proven a few basic properties of the real numbers. We can now finally describe some of the useful properties of the real numbers. The least upper bound property is the first such property. It will give a rigorous statement to the intuition that the real numbers "have no gaps" between them. We will see more rigorous statements of this intuition within our discussion of limits and completeness.

Definition 7.1 (**Upper bound**). Let E be a subset of $\mathbb{R}$, and let M be a real number. We say that M is an **upper bound** for E if and only if for every x in E, we have $x \leq M$.

Example 7.2. The set $\{t \in \mathbb{R} \colon 0 \leq t \leq 1\}$ has an upper bound of 1. The set $\{t \in \mathbb{R} \colon t > 0\}$ has no upper bound.

Definition 7.3 (**Least upper bound**). Let E be a subset of $\mathbb{R}$, and let M be a real number. We say that M is a **least upper bound** for E if and only if: M is an upper bound for E, and any other upper bound M' of E satisfies $M \leq M'$.

Example 7.4. The set $\{t \in \mathbb{R} \colon 0 \leq t \leq 1\}$ has a least upper bound of 1.

Proposition 7.5. *Let E be a subset of $\mathbb{R}$. Then E has at most one least upper bound.*

Proof. Let M, M' be two least upper bounds for E. We will show that $M = M'$. From Definition 7.3 applied to M, we have $M \leq M'$. From Definition 7.3 applied to M', we have $M' \leq M$. Therefore, $M = M'$. □

The following Theorem is taken as an axiom in the book. However, it can instead be proven from our construction of the real numbers. The proof is a bit long, so it could be skipped on a first reading.

Theorem 7.6 (**Least Upper Bound Property**). *Let E be a nonempty subset of $\mathbb{R}$. If E has some upper bound, then E has exactly one least upper bound.*

Proof. From Proposition 7.5, E has at most one least upper bound. We therefore need to show that E has at least one least upper bound. In order to find the least upper bound for E, we will construct a Cauchy sequence of rational numbers which come very close to the least upper bound of E.

Let M be an upper bound for E. Let $x_0 \in E$, and let n be a positive integer. From the Archimedean property (Theorem 6.36), there exists $K \in \mathbb{N}$ such that $x_0 + K/n > M$. That is, $x_0 + K/n$ is an upper bound for E. Since $x_0 \in E$, $x_0 - 1/n$ is not an upper bound for E. So, there exists an integer i with $0 \leq i \leq K$ such that $x_0 + i/n$ is an upper bound for E, though $x_0 + (i-1)/n$ is not an upper bound for E. To see that i exists, just let i be the smallest natural number such that $x_0 + i/n$ is an upper bound for E.

Note that $x_0 + (i-1)/n < x_0 + i/n$. From Corollary 6.37, there exists a rational number a_n such that

$$x_0 + (i-1)/n < a_n < x_0 + i/n.$$

Therefore, $a_n + 1/n$ is an upper bound for E since $a_n + 1/n > x_0 + i/n$, but $a_n - 1/n$ is not an upper bound for E since $a_n - 1/n < x_0 + (i-1)/n$.

Consider the sequence of rational numbers $(a_n)_{n=0}^{\infty}$. We will show that this sequence is a Cauchy sequence. Let n, m be positive integers. Then $a_n + 1/n$ is always an upper bound for E, while $a_m - 1/m$ is not an upper bound for E. Therefore, $a_n + 1/n > a_m - 1/m$. Similarly, $a_m + 1/m > a_n - 1/n$. Therefore, for all positive integers n, m,

$$-1/n - 1/m < a_n - a_m < 1/n + 1/m.$$

In particular, for any positive integer N, we have for all $n, m \geq N$,

$$-2/N < a_n - a_m < 2/N. \qquad (*)$$

Let $\varepsilon > 0$ be a rational number. From the Archimedean property (Theorem 6.36), there exists a positive integer N such that $N\varepsilon > 2$, so that $0 < 2/N < \varepsilon$. So, for any rational number ε, there exists a positive integer N such that, for all $n, m \geq N$, we have

$$-\varepsilon < a_n - a_m < \varepsilon.$$

So, $(a_n)_{n=0}^{\infty}$ is a Cauchy sequence.

Define $x := \mathrm{LIM}_{n\to\infty} a_n$. We will show that x is a least upper bound of E. We first show that x is an upper bound for E. Setting $m = N$ in $(*)$, we get that, for all $n \geq N$,

$$-2/N < a_n - a_N < 2/N.$$

So, from Proposition 6.30, for all positive integers N,

$$-2/N \leq x - a_N \leq 2/N. \qquad (**)$$

Let $y \in E$. For each positive integer N, recall that $a_N + 1/N$ is an upper bound for E. So, $y \leq a_N + 1/N$. From $(**)$, $-2/N \leq x - a_N$, so adding these two inequalities, we get $y - 3/N \leq x$. Since $y - 3/N \leq x$ for all positive integers N, we conclude that $y \leq x$. (Note that if we had $y > x$, then there exists a positive integer N such that $N(y - x) > 3$ by the Archimedean property, so $y - x > 3/N$, so $y - 3/N > x$, a contradiction.) In conclusion, x is an upper bound for E.

We now conclude by showing that x is the least upper bound for E. Let z be any other upper bound for E. We need to show that $x \leq z$. For any positive integer N, we know that $a_N - 1/N$ is not an upper bound for E. So, there exists $e \in E$ such that $a_N - 1/N < e \leq z$, so $a_N - 1/N < z$. From $(**)$, $x - a_N \leq 2/N$. Adding these two inequalities, $x < z + 3/N$ for all positive integers N. Therefore, $x \leq z$, as desired. $\square$

Definition 7.7 (**Supremum**)**.** Let E be a subset of $\mathbb{R}$ with some upper bound. The least upper bound of E is called the **supremum** of E. The supremum of E, which exists by Theorem 7.6, is denoted by $\sup(E)$ or $\sup E$. If E has no upper bound, we use the symbol $+\infty$ and we write $\sup(E) = +\infty$. If E is empty, we write $\sup(E) = -\infty$.

Definition 7.8 (**Infimum**)**.** Let E be a subset of $\mathbb{R}$ with some lower bound. The greatest lower bound of E is called the **infimum** of E. The infimum of E, which exists by Theorem 7.6, is denoted by $\inf(E)$ or $\inf E$. If E has no lower bound, we write $\inf(E) = -\infty$. If E is empty, we write $\inf(E) = +\infty$.

In Proposition 4.19, we saw that there does not exist a rational number x such that $x^2 = 2$. However, Theorem 7.6 allows us to show that there exists a real number x such that $x^2 = 2$. In this sense, the real numbers do not have a "gap" here. And indeed, we can always take the square root of a real positive number, and recover another positive real number.

Proposition 7.9. *There exists a real number x such that $x^2 = 2$.*

Proof. Let E be the set $E := \{y \in \mathbb{R} \colon y \geq 0 \,\text{and}\, y^2 < 2\}$. Note that E has an upper bound of 2, since $2^2 = 4 > 2$. So, by Theorem 7.6, there exists a real number x such that x is the unique least upper bound of E. We will show that $x^2 = 2$. In order to show $x^2 = 2$, we will show that either $x^2 < 2$ or $x^2 > 2$ lead to contradictions.

Assume for the sake of contradiction that $x^2 < 2$. Since 2 is an upper bound for E, and x is the least upper bound of E, we have $x \leq 2$. Let $0 < \varepsilon < 1$ be a real number. Then $\varepsilon^2 < \varepsilon$, so

$$(x + \varepsilon)^2 = x^2 + 2x\varepsilon + \varepsilon^2 \leq x^2 + 4\varepsilon + \varepsilon = x^2 + 5\varepsilon.$$

Since $x^2 < 2$, we can choose $0 < \varepsilon < 1$ such that $x^2 + 5\varepsilon < 2$, by the Archimedean property. That is, $(x + \varepsilon)^2 < 2$. So, $x + \varepsilon \in E$, but $x + \varepsilon > x$, contradicting the fact that x is an upper bound for E. We conclude that $x^2 < 2$ does not hold.

Now, assume for the sake of contradiction that $x^2 > 2$. As before, $1 \leq x \leq 2$. Let $0 < \varepsilon < 1$ be a real number. Then $\varepsilon^2 < \varepsilon$, so

$$(x - \varepsilon)^2 = x^2 - 2x\varepsilon + \varepsilon^2 \geq x^2 - 2x\varepsilon \geq x^2 - 4\varepsilon.$$

Since $x^2 > 2$, we can choose $0 < \varepsilon < 1$ such that $x^2 - 4\varepsilon > 2$, by the Archimedean property. That is, $(x - \varepsilon)^2 > 2$. So, for any $y \in E$, we must have $x - \varepsilon \geq y$. (If not, then $0 < x - \varepsilon < y$, so $(x - \varepsilon)^2 < y^2$, so $y^2 > 2$, contradicting that $y \in E$.) So, $x - \varepsilon$ is an upper bound for E, but $x - \varepsilon < x$, contradicting the fact that x is the least upper bound for E. We conclude that $x^2 > 2$ does not hold.

Finally, we conclude that $x^2 = 2$, as desired. □

8. Appendix: Notation

Let A, B be sets in a space X. Let m, n be a nonnegative integers. Let $\mathbb{F}$ be a field.

$\mathbb{Z} := \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$, the integers

$\mathbb{N} := \{0, 1, 2, 3, 4, 5, \ldots\}$, the natural numbers

$\mathbb{Z}_+ := \{1, 2, 3, 4, \ldots\}$, the positive integers

$\mathbb{Q} := \{m/n \colon m, n \in \mathbb{Z}, n \neq 0\}$, the rationals

$\mathbb{R}$ denotes the set of real numbers

$\mathbb{C} := \{x + y\sqrt{-1} : x, y \in \mathbb{R}\}$, the complex numbers

$\emptyset$ denotes the empty set, the set consisting of zero elements

$\in$ means "is an element of." For example, $2 \in \mathbb{Z}$ is read as "2 is an element of $\mathbb{Z}$."

$\forall$ means "for all"

$\exists$ means "there exists"

$\mathbb{F}^n := \{(x_1, \ldots, x_n) \colon x_i \in \mathbb{F}, \forall\, i \in \{1, \ldots, n\}\}$

$A \subseteq B$ means $\forall\ a \in A$, we have $a \in B$, so A is contained in B

$A \smallsetminus B := \{x \in A \colon x \notin B\}$

$A^c := X \smallsetminus A$, the complement of A

$A \cap B$ denotes the intersection of A and B

$A \cup B$ denotes the union of A and B

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