2: CARDINALITY, SEQUENCES, SERIES, SUBSEQUENCES

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1. Cardinality of Sets

In the previous sections, we constructed the real numbers, and discussed the completeness of the real numbers. We showed that the real numbers are a set of numbers that are larger than the rational numbers, in the sense that the rational numbers are contained in the real numbers. Also, there are real numbers that are not rational, such as the square root of two. There is even another sense in which the set of real numbers is much larger than the set of rational numbers. But what do we mean by this? There are evidently infinitely many rational numbers, and there are infinitely many real numbers. So how can one infinite thing be larger than another infinite thing? These questions lead us to the notion of cardinality.

The basic question we ask is: what does it mean for two sets to be of the same size? In essentially all cultures of the world, there are two fundamental concepts of numbers. The first concept is the notion of one, two and many. That is, essentially every culture of the world recognizes that the natural numbers exist, in some sense. (This is one reason that we call these numbers the natural numbers, after all.) The second concept of numbers is the notion of a bijective correspondence. What does it mean that I have the same number of apples and oranges? Well, it means that I can put the first apple next to the first orange, and I put the second apple next to the second orange, and so on, until every apple is matched to exactly one orange, and every orange is matched to exactly one apple. This is the notion of bijective correspondence which we use to define cardinality.

Let's now phrase this discussion using mathematical terminology. Let X, Y be sets, and let $f\colon X \to Y$ be a function.

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Definition 1.1 (**Bijection**)**.** The function $f\colon X \to Y$ is said to be **bijective** (or a **one-to-one correspondence**) if and only if: for every $y \in Y$, there exists exactly one $x \in X$ such that $f(x) = y$.

Example 1.2. Consider the sets $X = \{0, 1, 2\}$ and $Y = \{1, 2, 4\}$. Define $f\colon X \to Y$ by $f(0) = 1$, $f(1) = 4$ and $f(2) = 2$. Then f is a bijection.

Example 1.3. Consider the sets $X = \mathbb{N} = \{0, 1, 2, \ldots\}$ and $Y = \{1, 2, 3, 4, \ldots\}$. Define $f\colon X \to Y$ so that, for all $x \in X$, $f(x) := x + 1$. Then f is a bijection.

Remark 1.4. A function $f\colon X \to Y$ is bijective if and only if it is both injective and surjective. Also, if f is a bijection, then f is invertible. That is, there exists a function $f^{-1}\colon Y \to X$ such that $f(f^{-1}(y)) = y$ for all $y \in Y$, and $f^{-1}(f(x)) = x$ for all $x \in X$.

Definition 1.5 (**Cardinality**)**.** Two sets X, Y are said to have the same **cardinality** if and only if there exists a bijection from X onto Y.

Remark 1.6. The important thing to note here is that X and Y may be finite or infinite. At this point, it is not clear whether or not two infinite sets can have different cardinality. However, we will show below that the real numbers and the rational numbers do not have the same cardinality.

Exercise 1.7. Show that the notion of two sets having equal cardinality is an equivalence relation. That is, show:

- X has the same cardinality as X.
- If X has the same cardinality as Y, then Y has the same cardinality as X.
- If X has the same cardinality as Y, and if Y has the same cardinality as Z, then X has the same cardinality as Z.

Definition 1.8. Let n be a natural number. A set X is said to have **cardinality** n if and only if X has the same cardinality as $\{i \in \mathbb{N}\colon 1 \leq i \leq n\}$. We also say that X has n **elements** if and only if X has cardinality n.

Proposition 1.9. *Let n be a natural number, and suppose X is a set with cardinality n. Let m be any natural number such that $m \neq n$. Then X does not have cardinality m.*

Definition 1.10. A set X is **finite** if and only if there exists a natural number n such that X has cardinality n. Otherwise, the set X is called **infinite**.

Theorem 1.11. *The set of natural numbers $\mathbb{N}$ is infinite.*

Exercise 1.12. Using a proof by contradiction, prove Theorem 1.11.

Definition 1.13 (**Countable Set**)**.** A set X is said to be **countably infinite** (or just **countable**) if and only if X has the same cardinality as $\mathbb{N}$. A set X is said to be **at most countable** if X is either finite or countable.

Exercise 1.14. Let X be a subset of the natural numbers $\mathbb{N}$. Then X is at most countable.

Exercise 1.15. Let X be a subset of a countable set Y. Then X is at most countable.

Exercise 1.16. Let $f\colon \mathbb{N} \to Y$ be a function. Then $f(\mathbb{N})$ is at most countable. (Hint: consider the set $A := \{n \in \mathbb{N}\colon f(n) \neq f(m)\,\text{for all}\, 0 \leq m < n\}$. Prove that f is a bijection from A onto $f(\mathbb{N})$. Then use Exercise 1.14.)

Exercise 1.17. Let X be a countable set. Let $f\colon X \to Y$ be a function. Then $f(X)$ is at most countable.

We will now show that the integers and the rational numbers are countable.

Proposition 1.18. *Let X, Y be countable sets. Then $X \cup Y$ is a countable set.*

Exercise 1.19. Prove Proposition 1.18

Corollary 1.20. *The integers $\mathbb{Z}$ are countable.*

Proof. Write $\mathbb{Z} = \{0, 1, 2, \ldots\} \cup \{-1, -2, -3, \ldots\}$. We have therefore written $\mathbb{Z}$ as the union of two countable sets. Applying Proposition 1.18, we see that $\mathbb{Z}$ is countable. □

Definition 1.21 (Cartesian product). Let X, Y be sets. Define the set $X \times Y$ so that

$$X \times Y := \{(x, y)\colon x \in X \,\text{and}\, y \in Y\}.$$

The following strengthening of Proposition 1.18 shows that a countable union of countable sets is still countable.

Lemma 1.22. $\mathbb{N} \times \mathbb{N}$ *is countable.*

Proof. We need to construct a bijection $f\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$. Let $k \in \mathbb{N}$, and consider the "diagonal"

$$D_k := \{(x, y) \in \mathbb{N} \times \mathbb{N}\colon x + y = k\}.$$

Note that the cardinality of D_k is $k + 1$, and the cardinality of $D_0 \cup D_1 \cup \cdots \cup D_k$ is $1+2+\cdots+k+1 = (k+1)(k+2)/2$. Define $a_k := (k+1)(k+2)/2$. Note that $a_k+k+2 = a_{k+1}$. We define $f(0, 0) := 0$, and we then define f inductively as follows. Suppose we have defined f on $D_0, D_1, \ldots, D_k$ so that f maps $D_0 \cup D_1 \cup \cdots \cup D_k$ onto $\{0, 1, \ldots, a_k - 1\}$. Then, define $f(0, k + 1) := a_k$, $f(1, k) := a_k + 1$, $f(2, k - 1) := a_k + 2$, and so on. In general, for any $0 \leq j \leq k + 1$, define $f(j, k + 1 - j) := a_k + j$. We have therefore defined f so that f maps $D_0 \cup \cdots \cup D_{k+1}$ onto $\{0, 1, \ldots, a_{k+1} - 1\}$. The map f can be visualized in the following way

$$\begin{pmatrix} (0,0) & (0,1) & (0,2) & (0,3) & \cdots \\ (1,0) & (1,1) & (1,2) & \cdots & \\ (2,0) & (2,1) & \ddots & & \\ (3,0) & \vdots & & & \\ \vdots & & & & \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 1 & 3 & 6 & \cdots \\ 2 & 4 & 7 & \cdots & \\ 5 & 8 & \ddots & & \\ 9 & \vdots & & & \\ \vdots & & & & \end{pmatrix}$$

We now prove that f is a bijection. By the definition of f, if k is any natural number, then f is a bijection from D_k onto $\{a_k, a_k + 1, \ldots, a_{k+1} - 1\}$. We first show that f is injective. Let $(a, b), (c, d) \in \mathbb{N} \times \mathbb{N}$. Assume that $f(a, b) = f(c, d)$. For any natural numbers k, k' with $k \neq k'$, the sets of integers $\{a_k, a_k + 1, \ldots, a_{k+1} - 1\}$ and $\{a_{k'}, a_{k'} + 1, \ldots, a_{k'+1} - 1\}$ are disjoint. So, if $f(a, b) = f(c, d)$, there must exist a natural number k such that $f(a, b)$ and $f(c, d)$ are both contained in $\{a_k, a_k + 1, \ldots, a_{k+1} - 1\}$. Since f is a bijection from D_k onto $\{a_k, a_k + 1, \ldots, a_{k+1} - 1\}$, we conclude that $(a, b) = (c, d)$. Therefore, f is injective.

We now conclude by showing that f is surjective. Let $n \in \mathbb{N}$. We need to find $(a, b) \in \mathbb{N}\times\mathbb{N}$ such that $f(a, b) = n$. Since $\mathbb{N} = \cup_{k\in\mathbb{N}}\{a_k, a_k+1, \ldots, a_{k+1}-1\}$, there exists a natural number k such that n is in the set $\{a_k, a_k + 1, \ldots, a_{k+1} - 1\}$. Since f is a bijection from D_k onto $\{a_k, a_k + 1, \ldots, a_{k+1} - 1\}$, there exists $(a, b) \in D_k$ such that $f(a, b) = n$. Therefore, f is surjective. In conclusion, f is a bijection, as desired. □

Exercise 1.23. Using Lemma 1.22, prove the following statement. If X, Y are countable sets, then $X \times Y$ is countable.

Corollary 1.24. *The rational numbers $\mathbb{Q}$ are countable.*

Proof. From Corollary 1.20, the integers $\mathbb{Z}$ are countable. So, the nonzero integers $\mathbb{Z} \setminus \{0\}$ are also countable. Define a function $f\colon \mathbb{Z} \times (\mathbb{Z} \setminus \{0\}) \to \mathbb{Q}$ by

$$f(a,b) := a/b.$$

Since $b \neq 0$, f is well-defined. From Exercise 1.23, f is then a function from a countable set into the rational numbers $\mathbb{Q}$. Also, from the definition of the rational numbers, $f(\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})) = \mathbb{Q}$. From Exercise 1.17, we conclude that $\mathbb{Q}$ is at most countable. Since $\mathbb{Q}$ contains the integers, $\mathbb{Q}$ is not finite. Therefore, $\mathbb{Q}$ is countable, as desired. □

In summary, the natural numbers, integers, and rational numbers are countable. Surprisingly, the real numbers are not countable as we will show further below.

Definition 1.25 (Uncountable Set). Let X be a set. We say that X is **uncountable** if and only if X is not finite, and X is not countable.

Definition 1.26 (Power Set). Let X be a set. Define the **power set** 2^X as the set of all subsets of X. Equivalently, 2^X is the set of all functions $f\colon X \to \{0,1\}$.

Remark 1.27. To see the equivalence of these two definitions, for any subset A of X, we associate A with the function $f\colon X \to \{0,1\}$ where $f(x) = 1$ if and only if $x \in A$. In the other direction, given a function $f\colon X \to \{0,1\}$, we associate f to the set $A = \{x \in X : f(x) = 1\}$. This association gives a bijection between the subset of A, and the set of all functions $f\colon X \to \{0,1\}$.

Proposition 1.28. *Let X be a set. Then X and 2^X do not have the same cardinality.*

Proof. We argue by contradiction. Suppose X and 2^X have the same cardinality. Then there exists a bijection $f\colon X \to 2^X$. Consider the following subset V of X.

$$V := \{x \in X : x \notin f(x)\}.$$

We will achieve a contradiction by showing that V is not in the range of f. Since f is a bijection and $V \in 2^X$, there exists $y \in X$ such that $f(y) = V$. We now consider two cases.

Case 1. $y \in f(y)$. If $y \in f(y)$, then $y \in V$, since $f(y) = V$. However, from the definition of V, if $y \in V$, then $y \notin f(y)$, a contradiction.

Case 2. $y \notin f(y)$. If $y \notin f(y)$, then $y \notin V$, since $f(y) = V$. So, from the definition of V, $y \in f(y)$, a contradiction.

In either case, we get a contradiction. We conclude that X and 2^X do not have the same cardinality. □

Corollary 1.29. $\mathbb{N}$ *and $2^{\mathbb{N}}$ do not have the same cardinality. In particular, $2^{\mathbb{N}}$ is uncountable.*

Corollary 1.30. *The set of real numbers $\mathbb{R}$ is uncountable.*

Proof. Let $f\colon \mathbb{N} \to \{0,1\}$ be an element of $2^{\mathbb{N}}$. For any natural number n, define

$$a_n := \sum_{i=1}^{n} 3^{-i} f(i).$$

One can show that $(a_n)_{n=0}^{\infty}$ is a Cauchy sequence of rational numbers. We therefore define a map $F\colon 2^{\mathbb{N}} \to \mathbb{R}$ so that

$$F(f) := (\sum_{i=1}^{n} 3^{-i} f(i))_{n=0}^{\infty}.$$

We will show that F is an injection. Let $f, g\colon \mathbb{N} \to \{0,1\}$ such that $f \neq g$. Then there exists $N \in \mathbb{N}$ such that $f(N) \neq g(N)$. Without loss of generality, N is the smallest element of $\mathbb{N}$ such that $f(N) \neq g(N)$. Also, without loss of generality, $f(N) = 1$ and $g(N) = 0$. By the definition of N, we have $f(i) = g(i)$ for all $1 \leq i \leq N-1$. Therefore,

$$\begin{aligned} F(f) - F(g) &= (\sum_{i=1}^{n} 3^{-i} f(i))_{n=0}^{\infty} - (\sum_{i=1}^{n} 3^{-i} g(i))_{n=0}^{\infty} \\ &= (\sum_{i=1}^{n} 3^{-i} (f(i) - g(i)))_{n=0}^{\infty} = (3^{-N} + \sum_{i=N+1}^{n} 3^{-i} (f(i) - g(i)))_{n=N}^{\infty} \end{aligned}$$

Since $f(i), g(i) \in \{0,1\}$ for all $i \in \mathbb{N}$, we have $|f(i) - g(i)| \leq 1$. So, for any $n \geq N+1$, we have by the triangle inequality

$$\left| \sum_{i=N+1}^{n} 3^{-i} (f(i) - g(i)) \right| \leq \sum_{i=N+1}^{n} 3^{-i} \leq (2/3) 3^{-N}.$$

So, $3^{-N} + \sum_{i=N+1}^{n} 3^{-i}(f(i) - g(i)) \geq 3^{-N} - (2/3)3^{-N} = 3^{-N-1}$. Therefore, $F(f) - F(g) \geq 3^{-N-1} > 0$. In particular, $F(f) \neq F(g)$.

We conclude that $F\colon 2^{\mathbb{N}} \to \mathbb{R}$ is an injection. From Corollary 1.29, $2^{\mathbb{N}}$ is uncountable. Since F is an injection, F is a bijection onto its image $F(2^{\mathbb{N}})$. That is, $F(2^{\mathbb{N}})$ is uncountable. Finally, if $\mathbb{R}$ were countable, then all of its subsets would be at most countable, by Exercise 1.15. But we have found an uncountable subset $F(2^{\mathbb{N}})$ of $\mathbb{R}$. We therefore conclude that $\mathbb{R}$ is not countable. We also know that $\mathbb{R}$ is not finite, since it contains $\mathbb{N}$. We conclude that $\mathbb{R}$ is uncountable. □

2. Review

Theorem 2.1 (**Archimedean Property**). *Let x, ε be any positive real numbers. Then there exists a positive integer N such that $N\varepsilon > x$.*

Corollary 2.2. *Let x, z be real numbers with $x < z$. Then there exists a rational number y with $x < y < z$.*

Theorem 2.3 (**Least Upper Bound Property**). *Let E be a nonempty subset of $\mathbb{R}$. If E has some upper bound, then E has exactly one least upper bound.*

3. Sequences of Real Numbers

This course has a few fundamental concepts. One of these fundamental concepts is the Cauchy sequence. We will now introduce another fundamental concept, which is a variation on the Cauchy sequence. We will discuss sequences of real numbers and their limits. This topic is perhaps a bit more familiar, though it will turn out that a sequence of real numbers will have a limit if and only if this sequence is a Cauchy sequence. So, in some sense, we have been working with a familiar topic all along.

Our more general discussion of sequences of real numbers will inform our later investigation of derivatives and integration. More specifically, we can define derivatives and integrals in terms of limits of sequences of real numbers. So, a thorough understanding of limits of sequences of real numbers allows a quick and thorough investigation of derivatives and integrals.

Definition 3.1 (**Cauchy Sequence**)**.** Let $(a_n)_{n=0}^{\infty}$ be a sequence of real numbers. We say that $(a_n)_{n=0}^{\infty}$ is a **Cauchy sequence** if and only if, for any real $\varepsilon > 0$, there exists a natural number $N = N(\varepsilon)$ such that, for all $n, m \geq N$, we have $|a_n - a_m| < \varepsilon$.

Remark 3.2. Our previous definition of a Cauchy sequence asked for the same condition to hold for all rational $\varepsilon > 0$. So, Definition 3.1 may appear to be stricter than our previous definition of a Cauchy sequence. However, given any real $\varepsilon > 0$, Corollary 2.2 gives a rational $\varepsilon' > 0$ with $\varepsilon' < \varepsilon$. So, within Definition 3.1, it is equivalent to require the definition to hold for all rational $\varepsilon > 0$, or for all real $\varepsilon > 0$. That is, our previous definition and our current definition of a Cauchy sequence both coincide.

Definition 3.3 (**Convergent Sequence**)**.** Let $(a_n)_{n=0}^{\infty}$ be a sequence of real numbers, and let L be a real number. We say that the sequence $(a_n)_{n=0}^{\infty}$ **converges to** L if and only if, for every real $\varepsilon > 0$, there exists a natural number $N = N(\varepsilon)$ such that, for all $n \geq N$, we have $|a_n - L| < \varepsilon$.

Proposition 3.4. *Let $(a_n)_{n=0}^{\infty}$ be a sequence of real numbers, and let L, L' be a real numbers with $L \neq L'$. Then $(a_n)_{n=0}^{\infty}$ cannot simultaneously converge to L and converge to L'.*

Proof. We argue by contradiction. Suppose $(a_n)_{n=0}^{\infty}$ converges to L and to L'. Define $\varepsilon := |L - L'|/4 > 0$. Since $(a_n)_{n=0}^{\infty}$ converges to L, there exists N such that, for all $n \geq N$, we have $|a_n - L| < \varepsilon$. Since $(a_n)_{n=0}^{\infty}$ converges to L', there exists N' such that, for all $n \geq N'$, we have $|a_n - L'| < \varepsilon$. Setting $M := \max(N, N')$, we have

$$|a_M - L| < |L - L'|/4, \qquad |a_M - L'| < |L - L'|/4.$$

By the triangle inequality,

$$|L - L'| = |L - a_M + a_M - L'| \leq |a_M - L| + |a_M - L'| < |L - L'|/2.$$

Since $|L - L'| > 0$, we have shown that $2 < 1$, a contradiction. We conclude that it cannot occur that $(a_n)_{n=0}^{\infty}$ converges to L and to L' with $L \neq L'$. □

Definition 3.5 (**Limit**)**.** Let $(a_n)_{n=0}^{\infty}$ be a sequence of real numbers that is converging to a real number L. We then say that the sequence $(a_n)_{n=0}^{\infty}$ is **convergent**, and we write

$$L = \lim_{n\to\infty} a_n.$$

If $(a_n)_{n=0}^{\infty}$ is not convergent, we say that the sequence $(a_n)_{n=0}^{\infty}$ is **divergent**, and we say the limit of L is undefined.

Remark 3.6. By Proposition 3.4, if $(a_n)_{n=0}^{\infty}$ converges to some limit L, then this limit is unique. So, we call L the **limit** of the sequence $(a_n)_{n=0}^{\infty}$.

Remark 3.7. Instead of writing $(a_n)_{n=0}^{\infty}$ converges to L, we will sometimes write $a_n \to L$ as $n \to \infty$.

Proposition 3.8. *$\lim_{n\to\infty}(1/n) = 0$.*

Proof. Let $\varepsilon > 0$ be a real number. By the Archimedean property (Theorem 2.1), there exists a positive integer N such that $0 < 1/N < \varepsilon$. So, for all $n \geq N$, we have $|a_n - 0| = |a_n| = 1/n \leq 1/N < \varepsilon$. □

Exercise 3.9. Let $(a_n)_{n=m}^{\infty}$ be a sequence of real numbers converging to 0. Show that $(|a_n|)_{n=m}^{\infty}$ also converges to zero.

The following Theorem shows that Cauchy sequences and convergent sequences are the same thing. This Theorem also demonstrates that the real numbers are complete, in that a Cauchy sequence of real numbers converges to a real number. Note that the corresponding statement for the rational numbers is false. That is, a Cauchy sequence of rational numbers does not necessarily converge to a rational number. So, in this sense, the real numbers do not have any "holes," but the rational numbers do.

Theorem 3.10 (Completeness of $\mathbb{R}$). *Let $(a_n)_{n=0}^{\infty}$ be a sequence of real numbers. Then $(a_n)_{n=0}^{\infty}$ is convergent if and only if $(a_n)_{n=0}^{\infty}$ is a Cauchy sequence.*

Exercise 3.11. Prove Theorem 3.10. (Hint: Given a Cauchy sequence $(a_n)_{n=0}^{\infty}$, use that the rationals are dense in the real numbers to replace each real a_n by some rational a_n', so that $|a_n - a_n'|$ is small. Then, ensure that the sequence $(a_n')_{n=0}^{\infty}$ is a Cauchy sequence of rationals and that $(a_n')_{n=0}^{\infty}$ defines a real number which is the limit of the original sequence $(a_n)_{n=0}^{\infty}$.)

As a Corollary of Theorem 3.10, the formal limits of Cauchy sequences of rationals are actual limits. That is, we used a sensible notation for formal limits during our construction of the real number system.

Corollary 3.12. *Let $(a_n)_{n=0}^{\infty}$ be a Cauchy sequence of rational numbers. Then $(a_n)_{n=0}^{\infty}$ converges to* $\mathrm{LIM}_{n\to\infty} a_n$. *That is,*

$$\mathrm{LIM}_{n\to\infty} a_n = \lim_{n\to\infty} a_n.$$

Definition 3.13. Let M be a real number. A sequence $(a_n)_{n=0}^{\infty}$ is **bounded by** M if and only if $|a_n| \leq M$ for all $n \in \mathbb{N}$. We say that $(a_n)_{n=0}^{\infty}$ is **bounded** if and only if there exists a real number M such that $(a_n)_{n=0}^{\infty}$ is bounded by M.

Recall that any Cauchy sequence of rational numbers is bounded. The proof of this statement also shows that any Cauchy sequence of real numbers is bounded. So, from Theorem 3.10 we get the following.

Corollary 3.14. *Every convergent sequence is bounded.*

Theorem 3.15 (Limit Laws). *Let $(a_n)_{n=0}^{\infty}$, $(b_n)_{n=0}^{\infty}$ be convergent sequences. Let x, y be real numbers such that $x = \lim_{n\to\infty} a_n$, $y = \lim_{n\to\infty} b_n$.*

(i) *The sequence $(a_n + b_n)_{n=0}^{\infty}$ converges to $x + y$. That is,*

$$\lim_{n\to\infty}(a_n + b_n) = (\lim_{n\to\infty} a_n) + (\lim_{n\to\infty} b_n).$$

(ii) *The sequence $(a_n b_n)_{n=0}^{\infty}$ converges to xy. That is,*

$$\lim_{n\to\infty}(a_n b_n) = (\lim_{n\to\infty} a_n)(\lim_{n\to\infty} b_n).$$

(iii) *For any real number c, the sequence $(ca_n)_{n=0}^{\infty}$ converges to cx. That is,*

$$c \lim_{n\to\infty} a_n = \lim_{n\to\infty}(ca_n).$$

(iv) *The sequence* $(a_n - b_n)_{n=0}^{\infty}$ *converges to* $x - y$. *That is,*

$$\lim_{n\to\infty}(a_n - b_n) = (\lim_{n\to\infty} a_n) - (\lim_{n\to\infty} b_n).$$

(v) *Suppose* $x \neq 0$ *and there exists* m *such that* $a_n \neq 0$ *for all* $n \geq m$. *Then* $(a_n^{-1})_{n=m}^{\infty}$ *converges to* x^{-1}. *That is,*

$$\lim_{n\to\infty} a_n^{-1} = (\lim_{n\to\infty} a_n)^{-1}.$$

(vi) *Suppose* $x \neq 0$ *and there exists* m *such that* $a_n \neq 0$ *for all* $n \geq m$. *Then* $(b_n/a_n)_{n=m}^{\infty}$ *converges to* y/x. *That is,*

$$\lim_{n\to\infty}(b_n/a_n) = (\lim_{n\to\infty} b_n)/(\lim_{n\to\infty} a_n).$$

(vii) *Suppose* $a_n \geq b_n$ *for all* $n \geq 0$. *Then* $x \geq y$.

Exercise 3.16. Prove Theorem 3.15.

4. The Extended Real Number System

Now that we have defined limits, it is slightly more convenient to add two additional symbols to the real number system, namely $+\infty$ and $-\infty$.

Definition 4.1 (Extended Real Number System). The **extended real number system** $\mathbb{R}^*$ is the real line $\mathbb{R}$ with two additional elements $+\infty$ and $-\infty$. These two additional elements are distinct from each other, and these two elements are distinct from all other elements of the real line. So, $\mathbb{R}^* = \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$. An extended real number x is called **finite** if and only if x is a real number, and x is called **infinite** if and only if x is equal to $+\infty$ or $-\infty$. (Note that these notions of finite and infinite are similar to but distinct from our notions of finite and infinite sets.)

Definition 4.2 (Negation). The operation of negation is defined for any extended real number x by defining $-(+\infty) := -\infty$, and $-(-\infty) := +\infty$. And for any finite extended real number x, we use the usual operation of negation.

So, $-(-x) = x$ for any $x \in \mathbb{R}^*$. We can also extend the order on $\mathbb{R}$ to an order on $\mathbb{R}^*$.

Definition 4.3 (Order). Let x, y be extended real numbers. We say that x is less than or equal to y, and we write $x \leq y$, if and only if one of the following statements holds.

- x, y are real numbers, and $x \leq y$ as real numbers.
- $y = +\infty$.
- $x = -\infty$.

We say that $x < y$ if and only if $x \leq y$ and $x \neq y$. We sometimes write $y > x$ to indicate $x < y$, and we sometimes write $y \geq x$ to indicate $x \leq y$.

Remark 4.4. One can then check that this order on $\mathbb{R}^*$ satisfies the usual properties of order. Let $x, y, z \in \mathbb{R}^*$. Then

- $x \leq x$
- If $x \leq y$ and $y \leq x$ then $x = y$.
- If $x \leq y$ and $y \leq z$ then $x \leq z$.
- If $x \leq y$ then $-y \leq -x$.

Remark 4.5. It would be nice to extend other operations such as addition and multiplication to the extended real number system. However, doing so could introduce several inconsistencies within the various arithmetic operations. So, we will not extend other operations of arithmetic to $\mathbb{R}^*$. For example, it seems reasonable to define $1 + \infty = \infty$ and $2 + \infty = \infty$, but then $1 + \infty = 2 + \infty$, so the cancellation law no longer holds on $\mathbb{R}^*$.

One convenient property of the extended real number system is that the supremum and infimum operations are a bit easier to handle. In particular, the Theorem below can be stated succinctly, without explicitly reverting to different cases.

Definition 4.6 (Supremum). Let E be a subset of $\mathbb{R}^*$. We define the **supremum** $\sup(E)$ or **least upper bound** of E by the following conditions.

- If E is contained in $\mathbb{R}$ (so that $+\infty$ and $-\infty$ are not elements of E), then $\sup(E)$ is already defined.
- If E contains $+\infty$, define $\sup(E) := +\infty$.
- If E does not contain $+\infty$, and if E does contain $-\infty$, then $E \setminus \{-\infty\}$ is a subset of $\mathbb{R}$. So, we define $\sup(E) := \sup(E \setminus \{-\infty\})$.

Definition 4.7 (Infimum). Let E be a subset of $\mathbb{R}^*$. We define the **infimum** $\inf(E)$ or **greatest lower bound** of E by $\inf(E) := -(\sup(-E))$.

Theorem 4.8. *Let E be a subset of $\mathbb{R}^*$. Then the following statements hold.*

- *For every $x \in E$, we have $x \leq \sup(E)$ and $x \geq \inf(E)$.*
- *Let $M \in \mathbb{R}^*$ be an upper bound for E, so that $x \leq M$ for all $x \in E$. Then $\sup(E) \leq M$.*
- *Let $M \in \mathbb{R}^*$ be a lower bound for E, so that $x \geq M$ for all $x \in E$. Then $\inf(E) \geq M$.*

Exercise 4.9. Prove Theorem 4.8

Remark 4.10 (Limits and Infinity). Let $(a_n)_{n=0}^{\infty}$ be a sequence. If for all positive integers M, there exists N such that, for all $n \geq N$, we have $a_n > M$, we then write $\lim_{n\to\infty} a_n = +\infty$. In this case, we still say that the limit of the sequence does not exist. If for all negative integers M, there exists N such that, for all $n \geq N$, we have $a_n < M$, we then write $\lim_{n\to\infty} a_n = -\infty$. In this case, we still say that the limit of the sequence does not exist.

5. Suprema and Infima of Sequences

The extended real number system and Theorem 4.8 simplify our notation for suprema and infima of sets. One of the main motivations for suprema and infima is that they will aid our rigorous investigation of sequences of real numbers. That is, given a sequence of real numbers $(a_n)_{n=0}^{\infty}$, we will consider the suprema and infima of the *subset of real numbers*, $\{a_n : n \in \mathbb{N}\} \subseteq \mathbb{R}$.

Definition 5.1 (Suprema and infima of a sequence)**.** Let $(a_n)_{n=m}^{\infty}$ be a sequence of real numbers. Define $\sup(a_n)_{n=m}^{\infty}$ to be the supremum of the set $\{a_n : n \geq m,\, n \in \mathbb{N}\}$. Define $\inf(a_n)_{n=m}^{\infty}$ to be the infimum of the set $\{a_n : n \geq m,\, n \in \mathbb{N}\}$.

Example 5.2. For any $n \in \mathbb{N}$, let $a_n := (-1)^n$. Then $\sup(a_n)_{n=0}^{\infty} = 1$ and $\inf(a_n)_{n=0}^{\infty} = -1$.

Example 5.3. For any positive integer n, let $a_n := 1/n$. Then $\sup(a_n)_{n=1}^{\infty} = 1$ and $\inf(a_n)_{n=1}^{\infty} = 0$. Note that the infimum of the sequence $(a_n)_{n=1}^{\infty}$ is not actually a member of the sequence $(a_n)_{n=1}^{\infty}$.

Proposition 5.4. *Let $(a_n)_{n=m}^{\infty}$ be a sequence of real numbers. Let x be the extended real number $x := \sup(a_n)_{n=m}^{\infty}$. Then $a_n \leq x$ for all $n \geq m$. Also, for any $M \in \mathbb{R}^*$ which is an upper bound for $(a_n)_{n=m}^{\infty}$ (so that $a_n \leq M$ for all $n \geq m$), we have $x \leq M$. Finally, for any $y \in \mathbb{R}^*$ such that $y < x$, there exists at least one integer n with $n \geq m$ such that $y < a_n \leq x$.*

Exercise 5.5. Prove Proposition 5.4 using Theorem 4.8.

In Corollary 3.14, we saw that every convergent sequence is bounded. The converse of this statement is not true. The sequence $a_n = (-1)^n$ is bounded in absolute value by 1, but this sequence is not convergent. However, if we change the statement of the converse slightly, then it does become both true and quite useful. For example, we have the following.

Proposition 5.6. *Let $(a_n)_{n=m}^{\infty}$ be a bounded sequence of real numbers. Assume also that $(a_n)_{n=m}^{\infty}$ is monotone increasing. That is, $a_{n+1} \geq a_n$ for all $n \geq m$. Then the sequence $(a_n)_{n=m}^{\infty}$ is convergent. In fact,*

$$\lim_{n\to\infty} a_n = \sup(a_n)_{n=m}^{\infty}.$$

Exercise 5.7. Prove Proposition 5.6 using Proposition 5.4.

Remark 5.8. One can similarly show that a bounded monotone decreasing sequence $(a_n)_{n=m}^{\infty}$ (i.e. a sequence with $a_{n+1} \leq a_n$ for all $n \geq m$) is convergent.

Remark 5.9. A sequence $(a_n)_{n=m}^{\infty}$ is said to be **monotone** if and only if it is monotone increasing or monotone decreasing. If $(a_n)_{n=m}^{\infty}$ is monotone, then from Proposition 5.6 and Corollary 3.14, we see that $(a_n)_{n=m}^{\infty}$ converges if and only if $(a_n)_{n=m}^{\infty}$ is bounded.

6. Limsup, Liminf, and Limit Points

In order to understand the limits of sequences, it is helpful to first generalize our notion of a limit to the notion of a limit point. We then study this slightly generalized notion of a limit. We will use the limsup and liminf as upper and lower bounds on the set of limit points, respectively. Ultimately, if we for example want to prove that the limit of a sequence exists, it will sometimes be much easier to find upper and lower bounds on the set of limit points. Then, if we can show that the upper bound is equal to the lower bound, then we will have shown that the sequence is convergent.

Definition 6.1 (Limit Point). Let $(a_n)_{n=m}^{\infty}$ be a sequence of real numbers and let x be a real number. We say that x is a **limit point** of the sequence $(a_n)_{n=m}^{\infty}$ if and only if: for every real $\varepsilon > 0$, for every natural number $N \geq m$, there exists $n \geq N$ such that $|a_n - x| < \varepsilon$.

Proposition 6.2. *Let $(a_n)_{n=m}^{\infty}$ be a sequence of real numbers that converges to a real number x. Then x is a limit point of $(a_n)_{n=m}^{\infty}$. Moreover, x is the only limit point of $(a_n)_{n=m}^{\infty}$.*

Exercise 6.3. Prove Proposition 6.2.

Definition 6.4 (Limsup). Let $(a_n)_{n=m}^{\infty}$ be a sequence of real numbers. For any natural number n with $n \geq m$, define $b_n := \sup_{t\geq n} a_t$. Since the set $\{t \in \mathbb{N} \colon t \geq n+1\}$ is contained in the set $\{t \in \mathbb{N} \colon t \geq n\}$, we conclude that $b_{n+1} \leq b_n$ for all $n \geq m$. That is the sequence $(b_n)_{n=m}^{\infty}$ is monotone decreasing. We therefore define the **limit superior** by $\limsup_{n\to\infty} a_n := \lim_{n\to\infty} b_n$. The limit on the right either exists as a real number, or if the limit does not exist, we denote this limit with the extended real number $-\infty$. In summary, the following definition makes sense by Remark 5.9.

$$\limsup_{n\to\infty} a_n := \lim_{n\to\infty} \sup_{m\geq n} a_m.$$

Definition 6.5 (Liminf). Let $(a_n)_{n=m}^{\infty}$ be a sequence of real numbers. Reasoning as before, if we define $b_n := \inf_{m\geq n} a_m$, then $b_{n+1} \geq b_n$ for all $n \geq m$. So, the following definition of the **limit inferior** makes sense.

$$\liminf_{n\to\infty} a_n := \lim_{n\to\infty} \inf_{m\geq n} a_m.$$

Remark 6.6.

$$\limsup_{n\to\infty} a_n = \inf_{n\geq m} \sup_{t\geq n} a_t, \quad \text{and} \quad \liminf_{n\to\infty} a_n = \sup_{n\geq m} \inf_{t\geq n} a_t.$$

These identities follows from the monotonicity in n of the sequences $\sup_{t\geq n} a_t$ and $\inf_{t\geq n} a_t$, and Proposition 5.6

Proposition 6.7 (Properties of Limsup/Liminf). *Let $(a_n)_{n=m}^{\infty}$ be a sequence of real numbers. Let L^+ be the limit superior of this sequence, and let L^- be the limit inferior of this sequence. (Note that $L^+, L^- \in \mathbb{R}^*$.)*

(i) *For every $x > L^+$ there exists $N \geq m$ such that $a_n < x$ for all $n \geq N$. For every $y < L^-$ there exists $N \geq m$ such that $a_n > y$ for all $n \geq N$.*
(ii) *For every $x < L^+$ and for every $N \geq m$ there exists $n \geq N$ such that $a_n > x$. For every $y > L^+$ and for every $N \geq m$ there exists $n \geq N$ such that $a_n < y$.*
(iii) $\inf(a_n)_{n=m}^{\infty} \leq L^- \leq L^+ \leq \sup(a_n)_{n=m}^{\infty}$.
(iv) *If c is any limit point of $(a_n)_{n=m}^{\infty}$, then $L^- \leq c \leq L^+$.*
(v) *If L^+ is finite, then it is a limit point of $(a_n)_{n=m}^{\infty}$. If L^- is finite, then it is a limit point of $(a_n)_{n=m}^{\infty}$.*
(vi) *Let c be a real number. If $(a_n)_{n=m}^{\infty}$ converges to c, then $L^+ = L^- = c$. Conversely, if $L^+ = L^- = c$, then $(a_n)_{n=m}^{\infty}$ converges to c.*

Proof of (i). If $L^+ = +\infty$, there is nothing to prove. So, assume that $L^+ \neq +\infty$. Then $L^+ \in \mathbb{R} \cup \{-\infty\}$. Let $x > L^+$. From Remark 6.6, $L^+ = \inf_{n\geq m} \sup_{t\geq n} a_t$. From Proposition 5.4, there exists $n \geq m$ such that $x > \sup_{t\geq n} a_t$. Using Proposition 5.4 again, for all $t \geq n$, we have $x > a_t$, as desired. The second assertion follows similarly. □

Proof of (ii). If $L^+ = -\infty$, there is nothing to prove. So, assume that $L^+ \neq -\infty$. Then $L^+ \in \mathbb{R} \cup \{+\infty\}$. Let $x < L^+$. From Remark 6.6, $L^+ = \inf_{n\geq m} \sup_{t\geq n} a_t$. From Proposition 5.4, for all $n \geq m$ we have $x < \sup_{t\geq n} a_t$. Using Proposition 5.4 again, there exists $t \geq m$ such that $x < a_t$, as desired. The second assertion follows similarly. □

Exercise 6.8. Prove parts (iii)-(vi) of Proposition 6.7

Remark 6.9. Proposition 6.7(iv) and Definitions 6.4,6.5 say that, if L^+ and L^- are both finite, then they are the largest and smallest limit points of the sequence, respectively. Proposition 6.7(vi) shows that, to test whether or not a sequence converges, it suffices to compute the limit superior and limit inferior of the sequence.

Lemma 6.10 (Comparison Principle). *Let $(a_n)_{n=m}^{\infty}$, $(b_n)_{n=m}^{\infty}$ be sequences of real numbers. Assume that $a_n \leq b_n$ for all $n \geq m$. Then*

- $\sup(a_n)_{n=m}^\infty \leq \sup(b_n)_{n=m}^\infty$.
- $\inf(a_n)_{n=m}^\infty \leq \inf(b_n)_{n=m}^\infty$.
- $\limsup_{n\to\infty} a_n \leq \limsup_{n\to\infty} b_n$.
- $\liminf_{n\to\infty} a_n \leq \liminf_{n\to\infty} b_n$.

Exercise 6.11. Prove Lemma 6.10.

Corollary 6.12 (Squeeze Test/ Squeeze Theorem). *Let $(a_n)_{n=m}^\infty$, $(b_n)_{n=m}^\infty$, $(c_n)_{n=m}^\infty$ be sequences of real numbers such that there exists a natural number M such that, for all $n \geq M$,*

$$a_n \leq b_n \leq c_n.$$

Assume that $(a_n)_{n=m}^\infty$ and $(c_n)_{n=m}^\infty$ converge to the same limit L. Then $(b_n)_{n=m}^\infty$ converges to L.

Exercise 6.13. Prove Corollary 6.12 using Lemma 6.10.

6.1. **Exponentiation by Rationals.** For x, y real numbers, it would be nice to define x^y in some way. In the case that x is negative and y is e.g. $1/3$, defining x^y requires complex analysis. In this class, we will only be able to define x^y for positive real numbers x. To this end, in this section, we will let x be a positive real number, and we will define x^y for rational y.

Definition 6.14. Let $x > 0$ be a positive real number, and let $n \geq 1$ be a positive integer. We define the n^{th} **root of** x, and write $x^{1/n}$, by the formula

$$x^{1/n} := \sup\{y \in \mathbb{R} \colon y \geq 0 \text{ and } y^n \leq x\}.$$

For x a positive real number and n a positive integer, we now show that $x^{1/n}$ is finite.

Lemma 6.15. *Let $x > 0$ be a positive real number, and let $n \geq 1$ be a positive integer. Then the set $E := \sup\{y \in \mathbb{R} \colon y \geq 0$ and $y^n \leq x\}$ is nonempty and bounded from above. Consequently, $x^{1/n}$ is a real number by the Least Upper Bound property (Theorem 2.3).*

Proof. Since x is positive, $0 \in E$, so E is nonempty. We now show that E is bounded from above. We consider two cases: $x \leq 1$ and $x > 1$. In the first case, $x \leq 1$, and we claim that 1 is an upper bound for E. That is, if $y \in \mathbb{R}$ and $y \geq 0$ with $y^n \leq x \leq 1$, then $y \leq 1$. We prove this by contradiction. Suppose $y > 1$. Since $y > 1$, it follows by induction on n that $y^n > 1$ as well, contradicting that $y^n \leq 1$. We conclude that E is bounded above by 1 when $x \leq 1$. We now consider the case $x > 1$. We claim that x is an upper bound for E. That is, if $y \in \mathbb{R}$ and $y \geq 0$ with $y^n \leq x$, then $y \leq x$. We prove this by contradiction. Suppose $y > x$. Since $x > 1$, we have $y > x > 1$. If then follows by induction on n that $y^n > x$, contradicting that $y^n \leq x$. We conclude that E is bounded above by x when $x > 1$. Having exhausted all cases for $x > 0$, we are done. □

Lemma 6.16. *Let $x, y > 0$ be positive real numbers, and let $n, m \geq 1$ be positive integers.*

(i) *If $y = x^{1/n}$, then $y^n = x$.*
(ii) *If $y^n = x$, then $y = x^{1/n}$.*
(iii) *$x^{1/n}$ is a positive real number.*
(iv) *$x > y$ if and only if $x^{1/n} > y^{1/n}$.*
(v) *If $x > 1$ then $x^{1/n}$ decreases when n increases. If $x < 1$, then $x^{1/n}$ increases when n increases. If $x = 1$, then $x^{1/n} = 1$ for all positive integers n.*

(vi) $(xy)^{1/n} = x^{1/n}y^{1/n}$.
(vii) $(x^{1/n})^{1/m} = x^{1/(nm)}$.

Exercise 6.17. Prove Lemma 6.16.

Remark 6.18. Note the following cancellation law from Lemma 6.16(ii). If x, y are positive real numbers, and if $x^n = y^n$ for a positive integer n, then $x = y$. Note that the positivity of x, y is needed, since $(-3)^2 = 3^2$, but $3 \neq -3$.

Given a positive x and a rational number q, we can now define x^q. Due to the density of rational numbers within the real numbers, we therefore come very close to a general definition of x^y where y is real.

Definition 6.19 (Exponentiation to a Rational). Let $x > 0$ be a positive real number, and let q be a rational number. We now define x^q. Write $q = a/b$ where a is an integer, and b is a positive integer. We then define

$$x^q := (x^{1/b})^a.$$

We now show that this definition is well-defined.

Lemma 6.20. *Let a, a' be integers and let b, b' be positive integers such that $a/b = a'/b'$. Let x be a positive real number. Then $(x^{1/b})^a = (x^{1/b'})^{a'}$.*

Proof. We consider three cases: $a = 0$, $a < 0$, and $a > 0$. If $a = 0$, then we must have $a' = 0$ since $a/b = a'/b'$, so $(x^{1/b})^0 = 1 = (x^{1/b'})^0$, as desired.

If $a > 0$, then $a' > 0$ since $a/b = a'/b'$, and $a, b, b' > 0$. Define $y := x^{1/(ab')}$. Since $ab' = a'b$, we have $y = x^{1/(a'b)}$. From Lemma 6.16(vii), $y = (x^{1/b})^{1/a'} = (x^{1/b'})^{1/a}$. From Lemma 6.16(ii), we therefore have $y^{a'} = x^{1/b}$ and $y^a = x^{1/b'}$. So,

$$(x^{1/b'})^{a'} = (y^a)^{a'} = y^{aa'} = (y^{a'})^a = (x^{1/b})^a.$$

So, the case $a > 0$ is done. Finally, suppose $a < 0$. Then $a' < 0$ as well, so $-a$ and $-a'$ are positive. From the previous case, $(x^{1/b})^{-a} = (x^{1/b'})^{-a'}$. Taking the reciprocal of both sides completes the proof. □

Lemma 6.21. *Let $x, y > 0$ be positive real numbers, and let q, r be rational numbers.*
(i) *x^q is a positive real number.*
(ii) *$x^{q+r} = x^q x^r$ and $(x^q)^r = x^{qr}$.*
(iii) *$x^{-q} = 1/x^q$.*
(iv) *If $q > 0$, then $x > y$ if and only if $x^q > y^q$.*
(v) *If $x > 1$, then $x^q > x^r$ if and only if $q > r$. If $x < 1$, then $x^q > x^r$ if and only if $q < r$.*

Exercise 6.22. Prove Lemma 6.21.

6.2. **Some Standard Limits.** We can now compute some standard limits.

Remark 6.23. Let c be a real number. Then $\lim_{n\to\infty} c = c$.

Proposition 3.8 gives us the following.

Corollary 6.24. *For any positive integer k, we have $\lim_{n\to\infty} 1/(n^{1/k}) = 0$.*

Proof. From Lemma 6.16, $1/(n^{1/k})$ is decreasing in n and bounded below by 0. By Proposition 5.6 (for decreasing sequences bounded from below), there exists a real number $L \geq 0$ such that

$$L = \lim_{n\to\infty} 1/(n^{1/k}).$$

Taking both sides to the power k, and applying Theorem 3.15(ii) k times,

$$L^k = [\lim_{n\to\infty} 1/(n^{1/k})]^k = \lim_{n\to\infty} 1/(n^{k/k}) = \lim_{n\to\infty}(1/n) = 0.$$

The last equality follows from Proposition 3.8. Since $L^k = 0$, we know that L is not positive by Lemma 6.21(i). Since $L \geq 0$, we conclude that $L = 0$, as desired. □

Remark 6.25. By using the limit laws as in Corollary 6.24, it follows that, for any positive rational $q > 0$, we have $\lim_{n\to\infty} 1/(n^q) = 0$. Consequently, n^q does not converge as $n \to \infty$.

Exercise 6.26. Let $-1 < x < 1$. Then $\lim_{n\to\infty} x^n = 0$. Using the identity $(1/x^n)x^n = 1$ for $x > 1$, conclude that x^n does not converge as $n \to \infty$ for $x > 1$.

Lemma 6.27. *For any $x > 0$, we have $\lim_{n\to\infty} x^{1/n} = 1$.*

Exercise 6.28. Prove Lemma 6.27. (Hint: first, given any $\varepsilon > 0$, show that $(1+\varepsilon)^n$ has no real upper bound M, as $n \to \infty$. To prove this claim, set $x = 1/(1+\varepsilon)$ and use Exercise 6.26. Now, with this preliminary claim, show that for any $\varepsilon > 0$ and for any real M, there exists a positive integer n such that $M^{1/n} < 1+\varepsilon$. Now, use these two claims, and consider the cases $y > 1$ and $y < 1$ separately.)

7. Infinite Series

We will now begin our discussion of infinite series. One reason to care about infinite series is that Fourier analysis essentially reduces the study of certain functions to the study of infinite series. For another motivation, our study of infinite series is a precursor to the study of sequences of functions, and to the study of integrals. So, the study of infinite series provides a foundation for several other important topics.

We will briefly discuss finite series, and we will then move on to infinite series.

7.1. **Finite Series.**

Definition 7.1 (Finite Series/ Finite Sum). Let m, n be integers. Let $(a_i)_{i=m}^n$ be a finite sequence of real numbers. Define the **finite sum** $\sum_{i=m}^n a_i$ by the recursive formula

$$\begin{aligned} \sum_{i=m}^{n} a_i &:= 0 && \text{, if } n < m \\ \sum_{i=m}^{n+1} a_i &:= (\sum_{i=m}^{n} a_i) + a_{n+1} && \text{, if } n \geq m-1 \end{aligned}$$

Remark 7.2. To clarify the expressions we have used, a series is an expression of the form $\sum_{i=m}^n a_i$, and this series is equal to a real number, which is itself the sum of the series. The distinction between series and sum is not really important.

The following properties of summation can be proven by various inductive arguments.

Lemma 7.3. *Let $m \leq n < p$ be integers, and let $(a_i)_{i=m}^n, (b_i)_{i=m}^n$ be a sequences of real numbers, let k be an integer, and let c be a real number. Then*

- $$\sum_{i=m}^{n} a_i + \sum_{i=n+1}^{p} a_i = \sum_{i=m}^{p} a_i.$$
- $$\sum_{i=m}^{n} a_i = \sum_{j=m+k}^{n+k} a_{j-k}.$$
- $$\sum_{i=m}^{n}(a_i + b_i) = (\sum_{i=m}^{n} a_i) + (\sum_{i=m}^{n} b_i).$$
- $$\sum_{i=m}^{n}(ca_i) = c(\sum_{i=m}^{n} a_i).$$
- $$\left|\sum_{i=m}^{n} a_i\right| \leq \sum_{i=m}^{n} |a_i|.$$
- $$\textit{If } a_i \leq b_i \textit{ for all } m \leq i \leq n, \textit{ then } \sum_{i=m}^{n} a_i \leq \sum_{i=m}^{n} b_i.$$

Exercise 7.4. Prove Lemma 7.3.

We can also define sums over finite sets.

Definition 7.5. Let X be a finite set of cardinality $n \in \mathbb{N}$. Let $f \colon X \to \mathbb{R}$ be a function. We define $\sum_{x \in X} f(x)$ as follows. Let $g \colon \{1, 2, \ldots, n\} \to X$ be any bijection, which exists since X has cardinality n. We then define

$$\sum_{x \in X} f(x) := \sum_{i=1}^{n} f(g(i)).$$

Exercise 7.6. Show that this definition is well defined. That is, for any two bijections $g, h \colon \{1, 2, \ldots, n\} \to X$, we have $\sum_{i=1}^{n} f(g(i)) = \sum_{i=1}^{n} f(h(i))$.

Lemma 7.3 translates readily to sums over finite sets.

Lemma 7.7. (i) *If X is empty and if $f \colon X \to \mathbb{R}$ is a function, then*

$$\sum_{x \in X} f(x) = 0.$$

(ii) *If $X = \{x_0\}$ consists of a single element and if $f \colon X \to \mathbb{R}$ is a function, then*

$$\sum_{x \in X} f(x) = f(x_0).$$

(iii) *If X is a finite set, if $f \colon X \to \mathbb{R}$ is a function, and if $g \colon Y \to X$ is a bijection between sets X, Y, then*

$$\sum_{x \in X} f(x) = \sum_{y \in Y} f(g(y)).$$

(iv) *Let $m \leq n$ be integers, let $(a_i)_{i=m}^{n}$ be a sequence of real numbers, and let $X = \{i \in \mathbb{N} : m \leq i \leq n\}$. Then*

$$\sum_{i=m}^{n} a_i = \sum_{i \in X} a_i.$$

(v) *Let X, Y be disjoint finite sets (so $X \cap Y = \emptyset$). Let $f : X \cup Y \to \mathbb{R}$ be a function. Then*

$$\sum_{x \in X \cup Y} f(x) = (\sum_{x \in X} f(x)) + (\sum_{y \in Y} f(y)).$$

(vi) *Let X be a finite set, let $f : X \to \mathbb{R}$ and let $g : X \to \mathbb{R}$ be functions. Then*

$$\sum_{x \in X} (f(x) + g(x)) = (\sum_{x \in X} f(x)) + (\sum_{x \in X} g(x)).$$

(vii) *Let X be a finite set, let $f : X \to \mathbb{R}$ be a function, and let $c \in \mathbb{R}$. Then*

$$\sum_{x \in X} (cf(x)) = c(\sum_{x \in X} f(x)).$$

(viii) *Let X be a finite set, let $f : X \to \mathbb{R}$ and let $g : X \to \mathbb{R}$ be functions such that $f(x) \leq g(x)$ for all $x \in X$. Then*

$$\sum_{x \in X} f(x) \leq \sum_{x \in X} g(x).$$

(ix) *Let X be a finite set, and let $f : X \to \mathbb{R}$ be a function. Then*

$$\left| \sum_{x \in X} f(x) \right| \leq \sum_{x \in X} |f(x)|.$$

Exercise 7.8. Prove Lemma 7.7.

Lemma 7.9. *Let X, Y be finite sets. Let $f : (X \times Y) \to \mathbb{R}$ be a function. Then*

$$\sum_{x \in X} (\sum_{y \in Y} f(x, y)) = \sum_{(x,y) \in X \times Y} f(x, y).$$

Exercise 7.10. Prove Lemma 7.9 by induction on the size of X.

Corollary 7.11 (Fubini's Theorem for finite sets). *Let X, Y be finite sets, and let $f : X \times Y \to \mathbb{R}$ be a function. Then*

$$\sum_{x \in X} (\sum_{y \in Y} f(x, y)) = \sum_{(x,y) \in X \times Y} f(x, y) = \sum_{(y,x) \in Y \times X} f(x, y) = \sum_{y \in Y} (\sum_{x \in X} f(x, y)).$$

Proof. Lemma 7.9 gives the first and last equalities. For the remaining middle equality, note that $g : X \times Y \to Y \times X$ defined by $g(x, y) := (y, x)$ is a bijection. So, Lemma 7.7(iii) completes the proof. □

Remark 7.12. As we saw in the first homework, Corollary 7.11 is false for infinite sums. So, we can already see that more care is needed when we pass to infinite sums.

7.2. **Infinite Series.**

Definition 7.13 (Infinite Series). Let $(a_n)_{n=m}^{\infty}$ be a sequence of real numbers. An infinite series is any formal expression of the form

$$\sum_{n=m}^{\infty} a_n.$$

We sometimes write this series as

$$a_m + a_{m+1} + a_{m+2} + \cdots .$$

So far, we have only given a formal definition for the expression $\sum_{n=m}^{\infty} a_n$. The sum only makes sense as a real number via the following definition.

Definition 7.14 (Convergent Sum). Let $\sum_{n=m}^{\infty} a_n$ be a formal infinite series. For any integer $N \geq m$, define the N^{th} **partial sum** S_N of this series by $S_N := \sum_{n=m}^{N} a_n$. Note that S_N is a real number. If the sequence $(S_N)_{N=m}^{\infty}$ converges to some limit L as $N \to \infty$, then we say that the infinite series $\sum_{n=m}^{\infty} a_n$ is **convergent**, and this infinite series **converges to** L. We also write $L = \sum_{n=m}^{\infty} a_n$ and say that L is the **sum** of the infinite series $\sum_{n=m}^{\infty} a_n$. If the partial sums diverge, then we say that the infinite series $\sum_{n=m}^{\infty} a_n$ is **divergent**, and we do not assign any real number to the infinite series $\sum_{n=m}^{\infty} a_n$.

Proposition 7.15. *Let $\sum_{n=m}^{\infty} a_n$ be a formal series of real numbers. Then $\sum_{n=m}^{\infty} a_n$ converges if and only if: for every real number $\varepsilon > 0$, there exists an integer $N \geq M$ such that, for all $p, q \geq N$,*

$$\left| \sum_{n=p}^{q} a_n \right| < \varepsilon.$$

Exercise 7.16. Prove Proposition 7.15. (Hint: use Theorem 3.10).

Corollary 7.17 (Zero Test). *Let $\sum_{n=m}^{\infty} a_n$ be a formal series of real numbers. If $\sum_{n=m}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$. Note that the contrapositive says: if a_n does not converge to zero as $n \to \infty$, then $\sum_{n=m}^{\infty} a_n$ does not converge.*

Exercise 7.18. Using Proposition 7.15, prove Corollary 7.17.

Remark 7.19. The converse of Corollary 7.17 is false. For example, the series $\sum_{n=1}^{\infty} 1/n$ does not converge. On the other hand, as we will see below, the series $\sum_{n=1}^{\infty} (-1)^n/n$ does converge.

Definition 7.20 (Absolute Convergence). Let $\sum_{n=m}^{\infty} a_n$ be a formal series of real numbers. We say that the series $\sum_{n=m}^{\infty} a_n$ is **absolutely convergent** if and only if the series $\sum_{n=m}^{\infty} |a_n|$ is convergent. If a series is not absolutely convergent, then it is absolutely divergent.

Proposition 7.21. *Let $\sum_{n=m}^{\infty} a_n$ be a formal series of real numbers. If this series is absolutely convergent, then it is convergent. Moreover,*

$$\left| \sum_{n=m}^{\infty} a_n \right| \leq \sum_{n=m}^{\infty} |a_n| .$$

Exercise 7.22. Prove Proposition 7.21.

Proposition 7.23 (Alternating Series Test). *Let $(a_n)_{n=m}^\infty$ be a decreasing sequence of nonnegative real numbers. That is, $a_{n+1} \leq a_n$ and $a_n \geq 0$ for all $n \geq m$. Then the series $\sum_{n=m}^\infty (-1)^n a_n$ converges if and only if $a_n \to 0$ as $n \to \infty$.*

Proof. Suppose $\sum_{n=m}^\infty (-1)^n a_n$ converges. From the Zero Test (Corollary 7.17), we know that $(-1)^n a_n \to 0$ as $n \to \infty$. Therefore, $a_n \to 0$ as $n \to \infty$ as desired.

We now prove the converse. The idea is that looking only at even partial sums (or odd partial sums) reveals a monotonicity of the sequence. Suppose $\lim_{n\to\infty} a_n = 0$. Let $N \geq m$ and define $S_N := \sum_{n=m}^N (-1)^n a_n$. Note that

$$S_{N+2} = S_N + (-1)^{N+1} a_{N+1} + (-1)^{N+2} a_{N+2} = S_N + (-1)^{N+1}(a_{N+1} - a_{N+2}).$$

Recall that $a_{N+1} \geq a_{N+2}$. So, if N is odd, then $S_{N+2} \geq S_N$, and if N is even, $S_{N+2} \leq S_N$.

Suppose N is even. Then for any natural number k, $S_{N+2k} \leq S_N$. Also, $S_{N+2k+1} \geq S_{N+1} = S_N - a_{N+1}$, and $S_{N+2k+1} = S_{N+2k} - a_{N+2k+1} \leq S_{N+2k}$ since $a_{N+2k+1} \geq 0$. So, for any natural number k,

$$S_N - a_{N+1} \leq S_{N+2k+1} \leq S_{N+2k} \leq S_N.$$

In summary, for any integer $n \geq N$,

$$S_N - a_{N+1} \leq S_n \leq S_N.$$

Using the assumption $a_n \to 0$, if we are given any $\varepsilon > 0$, there exists a natural number N such that, for all $n > N$, we have $|a_n| < \varepsilon$, so that

$$S_N - \varepsilon \leq S_n \leq S_N.$$

That is, for any $\varepsilon > 0$, there exists a natural number N such that, for all $j, k > N$, we have $|S_j - S_k| < \varepsilon$. So, the sequence $(S_n)_{n=m}^\infty$ is a Cauchy sequence, and it therefore converges by Theorem 3.10. □

The following Proposition should be contrasted with Lemma 7.3. Note in particular the extra assumptions that are needed in the following statements.

Proposition 7.24.

- *Let $\sum_{n=m}^\infty a_n$ be a series of real numbers converging to x, and let $\sum_{n=m}^\infty b_n$ be a series of real numbers converging to y. Then $\sum_{n=m}^\infty (a_n + b_n)$ is a convergent series that converges to $x + y$. That is,*

$$\sum_{n=m}^\infty (a_n + b_n) = (\sum_{n=m}^\infty a_n) + (\sum_{n=m}^\infty b_n).$$

- *Let $\sum_{n=m}^\infty a_n$ be a series of real numbers converging to x, and let c be a real number. Then $\sum_{n=m}^\infty (c a_n)$ is a convergent series that converges to cx. That is,*

$$\sum_{n=m}^\infty (c a_n) = c(\sum_{n=m}^\infty a_n).$$

- *Let* $\sum_{n=m}^{\infty} a_n$ *be a series of real numbers, and let* k *be a natural number. If one of the two series* $\sum_{n=m}^{\infty} a_n$ *or* $\sum_{n=m+k}^{\infty} a_n$ *converges, then the other also converges, and we have*

$$\sum_{n=m}^{\infty} a_n = (\sum_{n=m}^{m+k-1} a_n) + (\sum_{n=m+k}^{\infty} a_n).$$

- *Let* $\sum_{n=m}^{\infty} a_n$ *be a series of real numbers converging to* x*, and let* k *be an integer. Then* $\sum_{n=m+k}^{\infty} a_{n-k}$ *also converges to* x*.*

Exercise 7.25. Prove Proposition 7.24.

Remark 7.26. From Proposition 7.24, changing any finite number of terms of a series does not affect the convergence of the series. We will therefore eventually de-emphasize the starting index of a series.

7.3. **Sums of Nonnegative Numbers.** From Proposition 7.21, if a series converges absolutely, then it also converges. In practice, we often show that a series converges by showing that it is absolutely convergent. Therefore, it is nice to have several ways to show whether or not a series is absolutely convergent. In other words, given a series of nonnegative numbers, it is desirable to verify its convergence. So, in this section, we will discuss series of nonnegative numbers.

Let $\sum_{n=m}^{\infty} a_n$ be a series of nonnegative real numbers. Since $a_n \geq 0$ for all $n \geq m$, the partial sums $S_N := \sum_{n=m}^{N} a_n$ are increasing. That is, $S_{N+1} \geq S_N$ for all integers $N \geq m$. From Remark 5.9, $(S_N)_{N=m}^{\infty}$ is convergent if and only if it has an upper bound M. We summarize this discussion as follows.

Proposition 7.27. *Let* $\sum_{n=m}^{\infty} a_n$ *be a formal series of nonnegative real numbers. Then this series is convergent if and only if there exists a real number* M *such that, for all integers* $N \geq m$*, we have*

$$\sum_{n=m}^{N} a_n \leq M.$$

Corollary 7.28 (**Comparison Test**)**.** *Let* $\sum_{n=m}^{\infty} a_n, \sum_{n=m}^{\infty} b_n$ *be formal series of real numbers. Assume that* $|a_n| \leq b_n$ *for all* $n \geq m$*. If* $\sum_{n=m}^{\infty} b_n$ *is convergent, then* $\sum_{n=m}^{\infty} a_n$ *is absolutely convergent. Moreover,*

$$\left|\sum_{n=m}^{\infty} a_n\right| \leq \sum_{n=m}^{\infty} |a_n| \leq \sum_{n=m}^{\infty} b_n.$$

Exercise 7.29. Prove Corollary 7.28.

Remark 7.30. The contrapositive of Corollary 7.28 says: if $|a_n| \leq b_n$ for all $n \geq m$, and if $\sum_{n=m}^{\infty} a_n$ is absolutely divergent, then $\sum_{n=m}^{\infty} b_n$ does not converge.

Example 7.31. Let x be a real number and consider the series

$$\sum_{n=0}^{\infty} x^n.$$

If $|x| \geq 1$, then this series diverges by the Zero Test (Corollary 7.17). If $|x| < 1$, then we can use induction to show that the partial sums satisfy

$$\sum_{n=0}^{N} x^n = (1 - x^{N+1})/(1 - x). \qquad (*)$$

If $|x| < 1$ then $\lim_{N\to\infty} x^N = 0$ by Exercise 6.26. So, using the Limit Laws,

$$\lim_{N\to\infty} (1 - x^{N+1})/(1 - x) = 1/(1 - x).$$

So, $\sum_{n=0}^{\infty} x^n$ converges to $1/(1 - x)$ when $|x| < 1$. Moreover, this convergence is absolute, by Corollary 7.28.

Proposition 7.32 (Dyadic Criterion). *Let $(a_n)_{n=1}^{\infty}$ be a decreasing sequence of nonnegative real numbers. That is, $a_{n+1} \leq a_n$ and $a_n \geq 0$ for all $n \geq 1$. Then the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the following series converges:*

$$\sum_{k=0}^{\infty} 2^k a_{(2^k)} = a_1 + 2a_2 + 4a_4 + 8a_8 + \cdots .$$

Proof. Let N be a positive integer and let K be a natural number. Let $S_N := \sum_{n=1}^{N} a_n$, and let $T_K := \sum_{k=0}^{K} 2^k a_{2^k}$. We claim that

$$S_{2^{K+1}-1} \leq T_K \leq 2S_{2^K}. \qquad (*)$$

We prove this claim by induction on K. In the case $K = 0$, we want to show $S_1 \leq T_0 \leq 2S_1$. Now, $S_1 = a_1$ and $T_0 = a_1$, so $S_1 \leq T_0 \leq 2S_1$ holds.

We now prove the inductive step. Suppose $(*)$ holds for some K. Then, note that

$$S_{2^{K+2}-1} = S_{2^{K+1}-1} + \sum_{n=2^{K+1}}^{2^{K+2}-1} a_n \leq S_{2^{K+1}-1} + \sum_{n=2^{K+1}}^{2^{K+2}-1} a_{2^{K+1}} = S_{2^{K+1}-1} + 2^{K+1} a_{2^{K+1}}.$$

Similarly,

$$S_{2^{K+1}} = S_{2^K} + \sum_{n=2^K+1}^{2^{K+1}} a_n \geq S_{2^K} + \sum_{n=2^K+1}^{2^{K+1}} a_{2^{K+1}} = S_{2^K} + 2^K a_{2^{K+1}}.$$

So, applying the inductive hypothesis,

$$S_{2^{K+2}-1} \leq T_K + 2^{K+1} a_{2^{K+1}} = T_{K+1}.$$
$$2S_{2^{K+1}} \geq T_K + 2^{K+1} a_{2^{K+1}} = T_{K+1}$$

So, we have completed the inductive step for $(*)$, thereby proving $(*)$.

We can now use $(*)$ to complete the proof. If $\sum_{n=1}^{\infty} a_n$ converges, then the partial sums S_{2^K} are bounded as $K \to \infty$ by Proposition 7.27. So the right inequality of $(*)$ shows that the partial sums T_K are bounded as $K \to \infty$. So, by Proposition 7.27, $\sum_{k=0}^{\infty} 2^k a_{(2^k)}$ converges. Conversely, suppose $\sum_{k=0}^{\infty} 2^k a_{(2^k)}$ converges. Then the partial sums T_K are bounded as $K \to \infty$ by Proposition 7.27. By $(*)$, the partial sums S_{2^K} are bounded as $K \to \infty$. Now, for any positive integer n, there exists a natural number N such that $n \leq 2^N$. So, since $S_n \leq S_{n+1}$ for all natural numbers n, we conclude that $S_n \leq S_{2^N}$. So, the partial sums S_n are bounded as $n \to \infty$. That is, $\sum_{n=1}^{\infty} a_n$ converges, by Proposition 7.27. □

Corollary 7.33. *Let $q > 0$ be a rational number. Then the series $\sum_{n=1}^{\infty} 1/n^q$ is convergent when $q > 1$ and it is divergent when $q \leq 1$.*

Proof. The sequence $(1/n^q)_{n=1}^{\infty}$ is nonnegative and decreasing by Lemma 6.21(iv). We can therefore apply the Dyadic Criterion (Theorem 7.32). The series $\sum_{n=1}^{\infty} 1/n^q$ is then convergent if and only if the following series is convergent

$$\sum_{k=0}^{\infty} 2^k \frac{1}{(2^k)^q} = \sum_{k=0}^{\infty} (2^{1-q})^k.$$

In the last equality, we used Lemma 6.21(ii). In Example 7.31, we showed that the geometric series $\sum_{k=0}^{\infty} x^k$ is convergent if and only if $|x| < 1$. So, the series $\sum_{n=1}^{\infty} 1/n^q$ is convergent if and only if $|2^{1-q}| < 1$, i.e. if and only if $q > 1$. (The last claim follows by Lemma 6.21.) □

Remark 7.34. In particular, the **harmonic series** $\sum_{n=1}^{\infty} 1/n$ diverges.

7.4. **Rearrangement of Series.** Let $(a_n)_{n=1}^{N}$ be a sequence of real numbers. From Exercise 7.6, any rearrangement of a finite series gives the same sum. That is, for any bijection $g\colon \{1, \ldots, n\} \to \{1, \ldots, n\}$, we have

$$\sum_{n=1}^{N} a_n = \sum_{n=1}^{N} a_{g(n)}.$$

The corresponding statement for infinite series is false. For example, consider the sequence $a_n = (-1)^{n+1}/(n+1)$. Recall that $\sum_{n=0}^{\infty} a_n$ converges by the Alternating Series Test (Proposition 7.23). However, there exists a bijection $g\colon \mathbb{N} \to \mathbb{N}$ such that $\sum_{n=0}^{\infty} a_n$ actually *diverges*. So, we cannot rearrange convergent infinite series and expect the sum of the rearranged series to be the same or even to converge at all.

Exercise 7.35. For any $n \in \mathbb{N}$, define $a_n := (-1)^{n+1}/(n+1)$. Find a bijection $g\colon \mathbb{N} \to \mathbb{N}$ such that the series $\sum_{n=0}^{\infty} a_{g(n)}$ diverges.

In fact, given any real number L, the series $\sum_{n=1}^{\infty}(-1)^n/n$ can be rearranged so that the rearranged series converges to L.

Theorem 7.36. *Let $\sum_{n=0}^{\infty} a_n$ be a convergent series which is not absolutely convergent. Let L be a real number. Then there exists a bijection $g\colon \mathbb{N} \to \mathbb{N}$ such that $\sum_{n=0}^{\infty} a_{g(n)}$ converges to L.*

However, we can rearrange absolutely convergent series.

Proposition 7.37. *Let $\sum_{n=0}^{\infty} a_n$ be an absolutely convergent series of real numbers. Let $g\colon \mathbb{N} \to \mathbb{N}$ be a bijection. Then $\sum_{m=0}^{\infty} a_{g(m)}$ is also convergent. Moreover,*

$$\sum_{n=0}^{\infty} a_n = \sum_{m=0}^{\infty} a_{g(m)}.$$

8. Ratio and Root Tests

The following test for series generalizes our investigation of the convergence of the geometric series from Example 7.31.

Theorem 8.1 (**Root Test**). *Let $\sum_{n=m}^{\infty} a_n$ be a series of real numbers. Define $\alpha := \limsup_{n\to\infty} |a_n|^{1/n}$.*

- *If $\alpha < 1$, then the series $\sum_{n=m}^{\infty} a_n$ is absolutely convergent. In particular, the series $\sum_{n=m}^{\infty} a_n$ is convergent.*
- *If $\alpha > 1$, then the series $\sum_{n=m}^{\infty} a_n$ is divergent.*
- *If $\alpha = 1$, no conclusion is asserted.*

Proof. First assume that $\alpha < 1$. Since $|a_n|^{1/n} \geq 0$ for every positive integer n, we know that $\alpha \geq 0$. Let $\varepsilon > 0$ so that $\varepsilon + \alpha < 1$. (For example, we could let $\varepsilon := (1-\alpha)/2$.) By Proposition 6.7(i), there exists an integer N such that, for all $n \geq N$, we have $|a_n|^{1/n} \leq (\alpha+\varepsilon)$. That is, $|a_n| \leq (\alpha+\varepsilon)^n$. Since $0 < \alpha+\varepsilon < 1$, the geometric series $\sum_{n=N}^{\infty}(\alpha+\varepsilon)^n$ converges. So, by the Comparison Test (Corollary 7.28), $\sum_{n=N}^{\infty} a_n$ converges. Therefore, $\sum_{n=m}^{\infty} a_n$ converges by Lemma 7.3, since a finite number of terms do not affect the convergence of the infinite sum.

Now, assume that $\alpha > 1$. By Proposition 6.7(ii), for every $N \geq m$ there exists $n \geq N$ such that $|a_n|^{1/n} \geq 1$. That is, $|a_n| \geq 1$. In particular, a_n does not converge to zero as $n \to \infty$. So, by the Zero Test (Corollary 7.17), we conclude that $\sum_{n=m}^{\infty} a_n$ does not converge. □

The Root Test is not always easy to use directly, but we can replace the roots by ratios, which are sometimes easier to handle.

Lemma 8.2. *Let $(b_n)_{n=m}^{\infty}$ be a sequence of positive numbers. Then*

$$\liminf_{n\to\infty} \frac{b_{n+1}}{b_n} \leq \liminf_{n\to\infty} b_n^{1/n} \leq \limsup_{n\to\infty} b_n^{1/n} \leq \limsup_{n\to\infty} \frac{b_{n+1}}{b_n}.$$

Proof. The middle inequality is Proposition 6.7(iii). We will only then prove the right inequality.

Let $L := \limsup_{n\to\infty} \frac{b_{n+1}}{b_n}$. If $L = +\infty$ there is nothing to show, so we assume that $L < +\infty$. Since b_n is positive for each $n \geq m$, we know that $L \geq 0$.

Let $\varepsilon > 0$. From Proposition 6.7(i), there exists an integer $N \geq m$ such that, for all $n \geq N$, we have $(b_{n+1}/b_n) \leq L+\varepsilon$. That is, $b_{n+1} \leq (L+\varepsilon)b_n$. By induction, we conclude that, for all $n \geq N$,

$$b_n \leq (L+\varepsilon)^{n-N} b_N.$$

That is, for all $n \geq N$,

$$b_n^{1/n} \leq (b_N(L+\varepsilon)^{-N})^{1/n}(L+\varepsilon). \qquad (*)$$

Letting $n \to \infty$ on the right side of $(*)$, and applying the Limit Laws and Lemma 6.27,

$$\lim_{n\to\infty} (b_N(L+\varepsilon)^{-N})^{1/n}(L+\varepsilon) = L+\varepsilon.$$

So, applying the Comparison Principle (Lemma 6.10 to $(*)$,

$$\limsup_{n\to\infty} b_n^{1/n} \leq L+\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $\limsup_{n\to\infty} b_n^{1/n} \leq L$, as desired. □

Exercise 8.3. Prove the left inequality of Lemma 8.2.

Combining Theorem 8.1 and Lemma 8.2 gives the following.

Corollary 8.4 (Ratio Test). *Let $\sum_{n=m}^{\infty} a_n$ be a series of nonzero numbers. (So, a_{n+1}/a_n is defined for any $n \geq m$.)*

- *If $\limsup_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} < 1$, then the series $\sum_{n=m}^{\infty} a_n$ is absolutely convergent. In particular, $\sum_{n=m}^{\infty} a_n$ is convergent.*
- *If $\liminf_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} > 1$, then the series $\sum_{n=m}^{\infty} a_n$ is divergent. In particular, $\sum_{n=m}^{\infty} a_n$ is not absolutely convergent.*

9. Subsequences

Our investigation now shifts attention from series back to sequences. We focus our attention on ways to decompose a sequence into smaller parts which are easier to understand. One popular paradigm in mathematics (and in science more generally) is to take a complicated object and break it into pieces which are simpler to understand. Subsequences are one manifestation of this paradigm.

Definition 9.1 (Subsequence). Let $(a_n)_{n=0}^{\infty}, (b_n)_{n=0}^{\infty}$ be sequences of real numbers. We say that $(b_n)_{n=0}^{\infty}$ is a **subsequence** of $(a_n)_{n=0}^{\infty}$ if and only if there exists a function $f \colon \mathbb{N} \to \mathbb{N}$ which is strictly increasing (i.e. $f(n+1) > f(n)$ for all $n \in \mathbb{N}$) such that, for all $n \in \mathbb{N}$,

$$b_n = a_{f(n)}$$

Example 9.2. The sequence $(a_{2n})_{n=0}^{\infty}$ is a subsequence of $(a_n)_{n=0}^{\infty}$, since $f(n) := 2n$ is an increasing function from $\mathbb{N}$ to $\mathbb{N}$, and $a_{2n} = a_{f(n)}$.

Here are some basic properties of subsequences.

Lemma 9.3. *Let $(a_n)_{n=0}^{\infty}, (b_n)_{n=0}^{\infty}, (c_n)_{n=0}^{\infty}$ be sequences of real numbers. Then $(a_n)_{n=0}^{\infty}$ is a subsequence of $(a_n)_{n=0}^{\infty}$. Also, if $(b_n)_{n=0}^{\infty}$ is a subsequence of $(a_n)_{n=0}^{\infty}$, and if $(c_n)_{n=0}^{\infty}$ is a subsequence of $(b_n)_{n=0}^{\infty}$, then $(c_n)_{n=0}^{\infty}$ is a subsequence of $(a_n)_{n=0}^{\infty}$.*

Exercise 9.4. Prove Lemma 9.3.

Subsequences and limits are closely related, as we now show.

Proposition 9.5. *Let $(a_n)_{n=0}^{\infty}$ be a sequence of real numbers, and let L be a real number.*

- *If the sequence $(a_n)_{n=0}^{\infty}$ converges to L, then every subsequence of $(a_n)_{n=0}^{\infty}$ converges to L.*
- *Conversely, if every subsequence of $(a_n)_{n=0}^{\infty}$ converges to L, then $(a_n)_{n=0}^{\infty}$ itself converges to L.*

Exercise 9.6. Prove Proposition 9.5.

Proposition 9.7. *Let $(a_n)_{n=0}^{\infty}$ be a sequence of real numbers, and let L be a real number.*

- *Suppose L is a limit point of $(a_n)_{n=0}^{\infty}$. Then there exists a subsequence of $(a_n)_{n=0}^{\infty}$ which converges to L.*
- *Conversely, if there exists a sequence of $(a_n)_{n=0}^{\infty}$ which converges to L, then L is a limit point of $(a_n)_{n=0}^{\infty}$.*

Exercise 9.8. Prove Proposition 9.7.

The following important theorem says: every bounded sequence has a convergent subsequence.

Theorem 9.9 (Bolzano-Weierstrass). *Let $(a_n)_{n=0}^{\infty}$ be a bounded sequence. That is, there exists a real number M such that $|a_n| \leq M$ for all $n \in \mathbb{N}$. Then there exists a subsequence of $(a_n)_{n=0}^{\infty}$ which converges.*

Proof. Let $L := \limsup_{n\to\infty} a_n$. From the Comparison Principle (Lemma 6.10), $|L| \leq M$. In particular, L is a real number. So, by Proposition 6.7(v), L is a limit point of $(a_n)_{n=0}^{\infty}$. By Proposition 9.7, there exists a subsequence of $(a_n)_{n=0}^{\infty}$ which converges to L. $\square$

Remark 9.10. Note that we could have defined $L := \liminf_{n\to\infty} a_n$ and the proof would have still worked.

10. Appendix: Notation

Let A, B be sets in a space X. Let m, n be a nonnegative integers.

$$
\begin{aligned}
\mathbb{Z} &:= \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}, \text{ the integers}\\
\mathbb{N} &:= \{0, 1, 2, 3, 4, 5, \ldots\}, \text{ the natural numbers}\\
\mathbb{Z}_+ &:= \{1, 2, 3, 4, \ldots\}, \text{ the positive integers}\\
\mathbb{Q} &:= \{m/n \colon m, n \in \mathbb{Z}, n \neq 0\}, \text{ the rationals}\\
\mathbb{R} &\text{ denotes the set of real numbers}\\
\mathbb{R}^* &= \mathbb{R} \cup \{-\infty\} \cup \{+\infty\} \text{ denotes the set of extended real numbers}\\
\mathbb{C} &:= \{x + y\sqrt{-1} : x, y \in \mathbb{R}\}, \text{ the complex numbers}\\
\emptyset &\text{ denotes the empty set, the set consisting of zero elements}\\
\in &\text{ means ``is an element of.'' For example, } 2 \in \mathbb{Z} \text{ is read as ``2 is an element of } \mathbb{Z}\text{.''}\\
\forall &\text{ means ``for all''}\\
\exists &\text{ means ``there exists''}\\
\mathbb{F}^n &:= \{(x_1, \ldots, x_n) \colon x_i \in \mathbb{F},\ \forall\, i \in \{1, \ldots, n\}\}\\
A \subseteq B &\text{ means } \forall\ a \in A, \text{ we have } a \in B, \text{ so } A \text{ is contained in } B\\
A \smallsetminus B &:= \{x \in A \colon x \notin B\}\\
A^c &:= X \smallsetminus A, \text{ the complement of } A\\
A \cap B &\text{ denotes the intersection of } A \text{ and } B\\
A \cup B &\text{ denotes the union of } A \text{ and } B
\end{aligned}
$$

Let E be a subset of $\mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$. Let $(a_n)_{n=0}^{\infty}$ be a sequence of real numbers.

$$\sup(E) \text{ denotes the smallest upper bound of } E$$
$$\inf(E) \text{ denotes the largest lower bound of } E$$
$$\limsup_{n\to\infty} a_n := \lim_{n\to\infty} \sup_{m\geq n} (a_n)_{n=m}^{\infty}$$
$$\liminf_{n\to\infty} a_n := \lim_{n\to\infty} \inf_{m\geq n} (a_n)_{n=m}^{\infty}$$

10.1. **Set Theory.** Let X, Y be sets, and let $f: X \to Y$ be a function. The function $f: X \to Y$ is said to be **injective** (or **one-to-one**) if and only if: for every $x, x' \in V$, if $f(x) = f(x')$, then $x = x'$.

The function $f: X \to Y$ is said to be **surjective** (or **onto**) if and only if: for every $y \in Y$, there exists $x \in X$ such that $f(x) = y$.

The function $f: X \to Y$ is said to be **bijective** (or a **one-to-one correspondence**) if and only if: for every $y \in Y$, there exists exactly one $x \in X$ such that $f(x) = y$. A function $f: X \to Y$ is bijective if and only if it is both injective and surjective.

Two sets X, Y are said to have the same **cardinality** if and only if there exists a bijection from X onto Y.

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