4: RIEMANN SUMS, RIEMANN INTEGRALS, FUNDAMENTAL THEOREM OF CALCULUS

STEVEN HEILMAN

Contents

1. Review

Theorem 1.1 (**Least Upper Bound Property**). *Let E be a nonempty subset of $\mathbb{R}$. If E has some upper bound, then E has exactly one least upper bound.*

Theorem 1.2. *Let $a < b$ be real numbers, and let $f\colon [a,b] \to \mathbb{R}$ be a function which is continuous on $[a,b]$. Then f is also uniformly continuous on $[a,b]$.*

Proposition 1.3. *Let X be a subset of $\mathbb{R}$, let x_0 be a limit point of X, let $f\colon X \to \mathbb{R}$ be a function, and let L be a real number. Then the following two statements are equivalent.*

- *f is differentiable at x_0 on X with derivative L.*
- *For every $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that, if $x \in X$ satisfies $|x - x_0| < \delta$, then*
$$|f(x) - [f(x_0) + L(x - x_0)]| \leq \varepsilon\, |x - x_0|\,.$$

Corollary 1.4 (**Mean Value Theorem**). *Let $a < b$ be real numbers, and let $f\colon [a,b] \to \mathbb{R}$ be a continuous function which is differentiable on (a,b). Then there exists $x \in (a,b)$ such that*
$$f'(x) = \frac{f(b) - f(a)}{b - a}.$$

Proposition 1.5. *Let X be a subset of $\mathbb{R}$, let x_0 be a limit point of X, and let $f\colon X \to \mathbb{R}$ be a function. If f is differentiable at x_0, then f is also continuous at x_0.*

Theorem 1.6 (**Properties of Derivatives**). *Let X be a subset of $\mathbb{R}$, let x_0 be a limit point of X, and let $f\colon X \to \mathbb{R}$ and $g\colon X \to \mathbb{R}$ be functions.*

(iv) *If f, g are differentiable at x_0, then fg is differentiable at x_0, and $(fg)'(x_0) = f'(x_0)g(x_0) + g'(x_0)f(x_0)$. **(Product Rule)***

Date: December 3, 2014.

2. Riemann Sums

Within calculus, the two most fundamental concepts are differentiation and integration. We have covered differentiation already, and we now move on to integration. Defining an integral is fairly delicate. In the case of the derivative, we created one limit, and the existence of this limit dictated whether or not the function in question was differentiable. In the case of the Riemann integral, there is also a limit to discuss, but it is much more complicated than in the case of differentiation.

We should mention that there is more than one way to construct an integral, and the Riemann integral is only one such example. Within this course, we will only be discussing the Riemann integral. The Riemann integral has some deficiencies which are improved upon by other integration theories. However, those other integration theories are more involved, so we focus for now only on the Riemann integral.

Our starting point will be partitions of intervals into smaller intervals, which will form the backbone of the Riemann sum. The Riemann sum will then be used to create the Riemann integral through a limiting constructing.

Definition 2.1 (**Partition**)**.** Let $a < b$ be real numbers. A **partition** P of the interval $[a, b]$ is a finite subset of real numbers $x_0, \ldots, x_n$ such that

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b.$$

We write $P = \{x_0, x_1, \ldots, x_n\}$.

Remark 2.2. Let P, P' be partitions of $[a, b]$. Then the union $P \cup P'$ of P and P' is also a partition of $[a, b]$.

Definition 2.3 (**Upper and Lower Riemann Sums**)**.** Let $a < b$ be real numbers, let $f\colon [a, b] \to \mathbb{R}$ be a bounded function, and let $P = \{x_0, \ldots, x_n\}$ be a partition of $[a, b]$. For every integer $1 \leq i \leq n$, the function $f|_{[x_{i-1},x_i]}$ is also a bounded function. So, $\sup_{x\in[x_{i-1},x_i]} f(x)$ and $\inf_{x\in[x_{i-1},x_i]} f(x)$ exist by the Least Upper Bound property (Theorem 1.1). We therefore define the **upper Riemann sum** $U(f, P)$ by

$$U(f, P) := \sum_{i=1}^{n} \Big(\sup_{x\in[x_{i-1},x_i]} f(x)\Big)(x_i - x_{i-1}).$$

We also define the **lower Riemann sum** $L(f, P)$ by

$$L(f, P) := \sum_{i=1}^{n} \Big(\inf_{x\in[x_{i-1},x_i]} f(x)\Big)(x_i - x_{i-1}).$$

Remark 2.4. For each integer $1 \leq i \leq n$, we define a function $g\colon [a, b] \to \mathbb{R}$ such that $g(x) := \sup_{y\in[x_{i-1},x_i]} f(y)$ for all $x_{i-1} \leq x < x_i$, with $g(b) := f(b)$. Then g is constant on $[x_{i-1}, x_i)$ for all $1 \leq i \leq n$, and $f(x) \leq g(x)$ for all $x \in [a, b]$. The upper Riemann sum $U(f, P)$ then represents the area under the function g, which is meant to upper bound the area under the function f. Similarly, for each integer $1 \leq i \leq n$, we define a function $h\colon [a, b] \to \mathbb{R}$ such that $h(x) := \inf_{y\in[x_{i-1},x_i]} f(y)$ for all $x_{i-1} \leq x < x_i$, with $h(b) := f(b)$. Then h is constant on $[x_{i-1}, x_i)$ for all $1 \leq i \leq n$, and $h(x) \leq g(x)$ for all $x \in [a, b]$. The lower Riemann sum $L(f, P)$ then represents the area under the function g, which is meant to lower bound the area under the function f.

Definition 2.5 (**Upper and Lower Integrals**). Let $a < b$ be real numbers, let $f\colon [a,b] \to \mathbb{R}$ be a bounded function. We define the **upper Riemann integral** $\overline{\int_a^b} f$ of f on $[a,b]$ by

$$\overline{\int_a^b} f := \inf\{U(f,P)\colon P \text{ is a partition of } [a,b]\}.$$

We also define the **lower Riemann integral** $\underline{\int_a^b} f$ of f on $[a,b]$ by

$$\underline{\int_a^b} f := \sup\{L(f,P)\colon P \text{ is a partition of } [a,b]\}.$$

Lemma 2.6. *Let $f\colon [a,b] \to \mathbb{R}$ be a bounded function, so that there exists a real number M such that $-M \leq f(x) \leq M$ for all $x \in [a,b]$. Then*

$$-M(b-a) \leq \underline{\int_a^b} f \leq \overline{\int_a^b} f \leq M(b-a)$$

In particular, $\overline{\int_a^b}$ and $\underline{\int_a^b}$ exist as real numbers.

Proof. If we choose P to be the partition $P = \{a,b\}$, then $U(f,P) = (b-a)\sup_{x\in[a,b]} f(x)$ and $L(f,P) = (b-a)\inf_{x\in[a,b]} f(x)$. So, $U(f,P) \leq (b-a)M$ and $L(f,P) \geq (b-a)(-M)$. So, $-M(b-a) \leq \underline{\int_a^b} f$ and $\overline{\int_a^b} f \leq M(b-a)$ by the definition of supremum and infimum, respectively.

We now show that $\underline{\int_a^b} f \leq \overline{\int_a^b} f$. Let P be any partition of $[a,b]$. By the definition of $L(f,P)$ and $U(f,P)$, we have $-\infty < L(f,P) \leq U(f,P) < +\infty$. So, we know that the set $\{U(f,P)\colon P \text{ is a partition of } [a,b]\}$ is nonempty and bounded from below. Similarly, the set $\{L(f,P)\colon P \text{ is a partition of } [a,b]\}$ is nonempty and bounded from above. Then, by the least upper bound property (Theorem 1.1), $\overline{\int_a^b} f$ and $\underline{\int_a^b} f$ exist as real numbers. So, given any $\varepsilon > 0$, choose a partition P such that $L(f,P) \geq \underline{\int_a^b} f - \varepsilon$. (Such a partition P exists by the definition of the supremum.) We then have $\underline{\int_a^b} f \leq L(f,P) + \varepsilon \leq U(f,P) + \varepsilon$. Taking the infimum over partitions P of $[a,b]$ of both sides of this inequality, we get $\underline{\int_a^b} f \leq \overline{\int_a^b} f + \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we conclude that $\underline{\int_a^b} f \leq \overline{\int_a^b} f$, as desired. □

3. Riemann Integral

Definition 3.1 (**Riemann Integral**). Let $a < b$ be real numbers, let $f\colon [a,b] \to \mathbb{R}$ be a bounded function. If $\overline{\int_a^b} f = \underline{\int_a^b} f$ we say that f is **Riemann integrable** on $[a,b]$, and we define

$$\int_a^b f := \overline{\int_a^b} f = \underline{\int_a^b} f.$$

Remark 3.2. Defining the Riemann integral of an unbounded function takes more care, and we defer this issue to later courses.

Theorem 3.3 (**Laws of integration**). *Let $a < b$ be real numbers, and let $f, g\colon [a,b] \to \mathbb{R}$ be Riemann integrable functions on $[a,b]$. Then*

(i) *The function $f + g$ is Riemann integrable on $[a,b]$, and $\int_a^b (f+g) = (\int_a^b f) + (\int_a^b g)$.*
(ii) *For any real number c, cf is Riemann integrable on $[a,b]$, and $\int_a^b (cf) = c(\int_a^b f)$.*
(iii) *The function $f - g$ is Riemann integrable on $[a,b]$, and $\int_a^b (f-g) = (\int_a^b f) - (\int_a^b g)$.*
(iv) *If $f(x) \geq 0$ for all $x \in [a,b]$, then $\int_a^b f \geq 0$.*
(v) *If $f(x) \geq g(x)$ for all $x \in [a,b]$, then $\int_a^b f \geq \int_a^b g$.*
(vi) *If there exists a real number c such that $f(x) = c$ for $x \in [a,b]$, then $\int_a^b f = c(b-a)$.*
(vii) *Let c, d be real numbers such that $c \leq a < b \leq d$. Then $[c,d]$ contains $[a,b]$. Define $F(x) := f(x)$ for all $x \in [a,b]$ and $F(x) := 0$ otherwise. Then F is Riemann integrable on $[c,d]$, and $\int_c^d F = \int_a^b f$.*
(viii) *Let c be a real number such that $a < c < b$. Then $f|_{[a,c]}$ and $f|_{[c,b]}$ are Riemann integrable on $[a,c]$ and $[c,b]$ respectively, and*

$$\int_a^b f = \int_a^c f|_{[a,c]} + \int_c^b f|_{[c,b]}.$$

Exercise 3.4. Prove Theorem 3.3.

Remark 3.5. Concerning Theorem 3.3(viii), we often write $\int_a^c f$ instead of $\int_a^c f|_{[a,c]}$.

3.1. **Riemann integrability of continuous functions.** So far we have discussed some properties of Riemann integrable functions, but we have not shown many functions that are actually Riemann integrable. In this section, we show that a continuous function on a closed interval is Riemann integrable.

Theorem 3.6. *Let $a < b$ be real numbers, and let $f\colon [a,b] \to \mathbb{R}$ be a continuous function on $[a,b]$. Then f is Riemann integrable.*

Proof. We will produce a family of partitions of the interval $[a,b]$ such that the upper and lower Riemann integrals of f are arbitrarily close to each other.

From Theorem 1.2, f is uniformly continuous on $[a,b]$. Let $\varepsilon > 0$. Then, by uniform continuity of f, there exists $\delta = \delta(\varepsilon) > 0$ such that, if $x, y \in [a,b]$ satisfy $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon$. By the Archimedean property, there exists a positive integer N such that $(b-a)/N < \delta$.

Consider the partition P of the interval $[a,b]$ of the form

$$P = \{x_0, \ldots, x_N\} = \{a, a+(b-a)/N, a+2(b-a)/N, a+3(b-a)/N, \ldots, a+(N-1)(b-a)/N, b\}.$$

Note that $x_i - x_{i-1} = (b-a)/N$ for all $1 \leq i \leq N$. Since f is continuous on $[a,b]$, f is also continuous on $[x_{i-1}, x_i]$ for each $1 \leq i \leq N$. In particular, $f|_{[x_{i-1},x_i]}$ achieves its maximum and minimum for all $1 \leq i \leq N$. So, for each $1 \leq i \leq N$, there exist $m_i, M_i \in [x_{i-1}, x_i]$ such that

$$\inf_{x \in [x_{i-1},x_i]} f(x) = f(m_i), \quad \text{and} \quad \sup_{x \in [x_{i-1},x_i]} f(x) = f(M_i).$$

Since $x_i - x_{i-1} = (b-a)/N < \delta$, we have $|m_i - M_i| < \delta$ for each $1 \leq i \leq n$. Since f is uniformly continuous, we conclude that

$$\inf_{x \in [x_{i-1},x_i]} f(x) = f(m_i) > f(M_i) - \varepsilon = (\sup_{x \in [x_{i-1},x_i]} f(x)) - \varepsilon, \quad \forall\, 1 \leq i \leq n. \qquad (*)$$

We now estimate $U(f,P)$ and $L(f,P)$. By the definition of $U(f,P)$ and $L(f,P)$, we have

$$L(f,P) \leq U(f,P). \qquad (**)$$

However, $L(f,P)$ is also close to $U(f,P)$ by $(*)$:

$$L(f,P) = \frac{b-a}{N}\sum_{i=1}^{N}(\inf_{x\in[x_{i-1},x_i]} f(x)) > \frac{b-a}{N}\sum_{i=1}^{N}[(\sup_{x\in[x_{i-1},x_i]} f(x)) - \varepsilon] = -(b-a)\varepsilon + U(f,P).$$

By the definition of $\underline{\int_a^b} f$, we conclude that

$$\underline{\int_a^b} f > -(b-a)\varepsilon + U(f,P).$$

By the definition of $\overline{\int_a^b} f$, we conclude that

$$\underline{\int_a^b} f > -(b-a)\varepsilon + \overline{\int_a^b} f.$$

Since $\varepsilon > 0$ is arbitrary, we get

$$\underline{\int_a^b} f \geq \overline{\int_a^b} f.$$

Combining this inequality with Lemma 2.6, we conclude that $\underline{\int_a^b} f = \overline{\int_a^b} f$. That is, f is Riemann integrable. □

Exercise 3.7. Let $a < b$ be real numbers. Let $f\colon [a,b] \to \mathbb{R}$ be a bounded function. Let $c \in [a,b]$. Assume that, for each $\delta > 0$, we know that f is Riemann integrable on the set $\{x \in [a,b]\colon\ |x-c| \geq \delta\}$. Then f is Riemann integrable on $[a,b]$.

3.2. **Piecewise Continuous Functions.** We can now expand a bit more the family of functions that are Riemann integrable.

Proposition 3.8. *Let $a < b$ be real numbers. Assume that $f\colon [a,b] \to \mathbb{R}$ is continuous at every point of $[a,b]$, except for a finite number of points. Then f is Riemann integrable.*

Proof. By Theorem 3.3(viii) and an inductive argument, it suffices to consider the case that f is discontinuous at a single point $c \in [a,b]$. Let $\delta > 0$. Then f is continuous on the set $E := \{x \in [a,b]\colon\ |x-c| \geq \delta\}$. Note that E consists of either one or two closed intervals. Since $f|_E$ is continuous, we then conclude that $f|_E$ Riemann integrable by Theorem 3.6. Then Exercise 3.7 says that f is Riemann integrable on $[a,b]$, as desired. □

3.3. **Monotone Functions.** It turns out that monotone functions are Riemann integrable as well. There exist monotone functions that are not piecewise continuous, so the current section is not subsumed by the previous one.

Proposition 3.9. *Let $a < b$ be real numbers, and let $f\colon [a,b] \to \mathbb{R}$ be a monotone function. Then f is Riemann integrable.*

Proof. Let $\varepsilon > 0$. Without loss of generality, f is monotone increasing. Then $f(a) \leq f(x) \leq f(b)$ for all $x \in [a,b]$, so f is bounded. By the Archimedean property, there exists a positive integer N such that $(b-a)(f(b)-f(a))/N < \varepsilon$.

Consider the partition P of the interval $[a,b]$ of the form

$$P = \{x_0, \ldots, x_N\} = \{a, a+(b-a)/N, a+2(b-a)/N, a+3(b-a)/N, \ldots, a+(N-1)(b-a)/N, b\}.$$

Note that $x_i - x_{i-1} = (b-a)/N$ for all $1 \leq i \leq N$. We now estimate $U(f,P)$ and $L(f,P)$. By the definition of $U(f,P)$ and $L(f,P)$, we have

$$L(f,P) \leq U(f,P). \qquad (*)$$

However, since f is monotonically increasing,

$$\begin{aligned} L(f,P) &= \frac{b-a}{N}\sum_{i=1}^{N}(\inf_{x\in[x_{i-1},x_i]} f(x)) \geq \frac{b-a}{N}\sum_{i=1}^{N} f(x_{i-1}) = \frac{b-a}{N}\left(f(x_0) + \sum_{i=1}^{N-1} f(x_i)\right) \\ &\geq \frac{b-a}{N}\left(f(x_0) + \sum_{i=1}^{N-1}(\sup_{x\in[x_{i-1},x_i]} f(x))\right) = \frac{b-a}{N}(f(x_0) - \sup_{x\in[x_{N-1},x_N]} f(x)) + U(f,P) \\ &\geq \frac{b-a}{N}(f(a)-f(b)) + U(f,P) \geq -\varepsilon + U(f,P). \end{aligned}$$

By the definition of $\underline{\int_a^b} f$, we conclude that

$$\underline{\int_a^b} f \geq -\varepsilon + U(f,P).$$

By the definition of $\overline{\int_a^b} f$, we conclude that

$$\underline{\int_a^b} f \geq -\varepsilon + \overline{\int_a^b} f.$$

Since $\varepsilon > 0$ is arbitrary, we get

$$\underline{\int_a^b} f \geq \overline{\int_a^b} f.$$

Combining this inequality with Lemma 2.6, we conclude that $\underline{\int_a^b} f = \overline{\int_a^b} f$. That is, f is Riemann integrable. □

3.4. **A Non-Riemann Integrable Function.** Unfortunately, not every function is Riemann integrable. We have seen that unbounded functions cause some difficulty in our definition of the Riemann integral, since their Riemann sums can be $+\infty$ or $-\infty$. However, there are even bounded functions that are not Riemann integrable.

Consider the following function $f\colon \mathbb{R} \to [0,1]$, which we encountered in our investigation of limits.

$$f(x) := \begin{cases} 1 & \text{, if } x \in \mathbb{Q} \\ 0 & \text{, if } x \notin \mathbb{Q} \end{cases}.$$

For any partition P of $[0,1]$, we automatically have $L(f,P)=0$ and $U(f,P)=1$. (Justify this statement.) Therefore, $\underline{\int_0^1} f = 0$ and $\overline{\int_0^1} f = 1$, so that this function f is not Riemann integrable on $[0,1]$.

4. Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus says, roughly speaking, that differentiation and integration negate each other. This fact is remarkable on its own, but it will also allow us to actually compute a wide range of integrals. (Note that we have not yet been able to compute any integrals.)

Theorem 4.1 (**First Fundamental Theorem of Calculus**)**.** *Let $a<b$ be real numbers. Let $f\colon [a,b]\to\mathbb{R}$ be a continuous function on $[a,b]$. Assume that f is also differentiable on $[a,b]$, and f' is Riemann integrable on $[a,b]$. Then*

$$\int_a^b f' = f(b)-f(a).$$

Proof. Let $P=\{x_0,\ldots,x_n\}$ be a partition of $[a,b]$. Then

$$f(b)-f(a) = f(x_n)-f(x_0) = \sum_{i=1}^{n}(f(x_i)-f(x_{i-1})). \qquad (*)$$

By the Mean Value Theorem (Corollary 1.4), for each $1\leq i\leq n$ there exists $y_i\in[x_{i-1},x_i]$ such that

$$(x_i-x_{i-1})f'(y_i) = f(x_i)-f(x_{i-1}).$$

Substituting these equalities into $(*)$, we get

$$f(b)-f(a) = \sum_{i=1}^{n}(x_i-x_{i-1})f'(y_i).$$

Applying the definitions of $L(f',P)$ and $U(f',P)$, we have

$$L(f',P)\leq f(b)-f(a)\leq U(f',P).$$

From Definition 2.5, we get

$$\underline{\int_a^b} f' \leq f(b)-f(a) \leq \overline{\int_a^b} f'. \qquad (**)$$

Since f' is Riemann integrable, $\underline{\int_a^b} f' = \overline{\int_a^b} f' = \int_a^b f'$. So, $(**)$ implies that $\int_a^b f' = f(b)-f(a)$, as desired. $\square$

Theorem 4.2 (**Second Fundamental Theorem of Calculus**)**.** *Let $a<b$ be real numbers. Let $f\colon[a,b]\to\mathbb{R}$ be a Riemann integrable function. Define a function $F\colon[a,b]\to\mathbb{R}$ by*

$$F(x) := \int_a^x f.$$

Then F is continuous. Moreover, if $x_0\in[a,b]$ and if f is continuous at x_0, then F is differentiable at x_0 and $F'(x_0)=f(x_0)$.

Proof. Since $f\colon [a,b] \to \mathbb{R}$ is Riemann integrable, f is bounded by the definition of Riemann integrability. So, there exists a real number M such that $-M \leq f(x) \leq M$ for all $x \in [a,b]$. Let $x, y \in [a,b]$. Without loss of generality, $x \leq y$. Then, by Theorem 3.3(viii)

$$F(y) - F(x) = \int_a^y f - \int_a^x f = \int_x^y f. \qquad (*)$$

So, by Theorem 3.3(v),

$$-M(y-x) \leq \int_x^y f = F(y) - F(x) = \int_x^y f \leq M(y-x).$$

That is, $|F(y) - F(x)| \leq M\,|y-x|$. Interchanging the roles of x and y leaves this statement unchanged, so for any $x, y \in [a,b]$, we have

$$|F(y) - F(x)| \leq M\,|y-x|\,.$$

In particular, F is uniformly continuous, so F is continuous.

Now, suppose f is continuous at x_0. Using Proposition 1.3, it suffices to show: there exists a real number L such that, for any $\varepsilon > 0$, there exists $\delta > 0$ such that, if $y \in [a,b]$ satisfies $|y - x_0| < \delta$, then

$$|F(y) - [F(x_0) + L(y - x_0)]| \leq \varepsilon\,|y - x_0|\,. \qquad (**)$$

We set $L := f(x_0)$. Let $\varepsilon > 0$. Applying the continuity of f at x_0, there exists $\delta > 0$ such that if $y \in [a,b]$ satisfies $|y - x_0| < \delta$, then

$$f(x_0) - \varepsilon \leq f(y) \leq f(x_0) + \varepsilon.$$

Assume first that y satisfies $y > x_0$. Then integrating and applying Theorem 3.3(v),

$$(f(x_0) - \varepsilon)(y - x_0) \leq \int_{x_0}^y f \leq (f(x_0) + \varepsilon)(y - x_0).$$

So, using $(*)$,

$$|F(y) - F(x_0) - f(x_0)(y - x_0)| = \left|\left(\int_{x_0}^y f\right) - f(x_0)(y - x_0)\right| \leq \varepsilon\,|y - x_0|\,.$$

That is, we proved $(**)$ holds for $y > x_0$. The case $y < x_0$ is proven similarly, and the case $y = x_0$ follows since then both sides of $(**)$ are zero. □

4.1. **Consequences of the Fundamental Theorem.** One of the consequences of the Fundamental Theorem of Calculus is that we can now actually compute some integrals. For example, if $\alpha \in \mathbb{Q}$, $\alpha \neq -1$, and if $0 < a < b$ are real numbers, then $f(x) := (\alpha+1)^{-1}x^{\alpha+1}$ satisfies $f'(x) = x^{\alpha}$. So, by Theorem 4.1,

$$\int_a^b x^{\alpha} = \frac{1}{\alpha+1}(b^{\alpha+1} - a^{\alpha+1}).$$

Remark 4.3. Let $\beta \in \mathbb{Q}$, let $x > 0$ and let $f(x) := x^{\beta}$. Let's justify the formula $f'(x) = \beta x^{\beta-1}$. Write $\beta = p/q$ with $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $q \neq 0$. Then $f(x) = (x^p)^{1/q}$. Recall that the function $h(x) = x^{1/q}$ is differentiable for $x > 0$ by the Inverse Function Theorem. If $p \geq 0$, then we have already verified by explicit calculation that $g(x) := x^p$ is differentiable. If $p < 0$, then $g(x) := x^p = 1/x^{-p}$ is differentiable by the quotient rule. In summary, we can write $f(x) = h(g(x))$, where h is differentiable when $g(x) > 0$, g is differentiable when $x > 0$, and $g(x) > 0$ when $x > 0$. So, f is differentiable when $x > 0$, by the chain rule.

Theorem 4.4. *Let $a < b$ be real numbers. Let $f, g\colon [a,b] \to \mathbb{R}$ be Riemann integrable functions. Then the product fg is Riemann integrable.*

Theorem 4.5 (Integration by Parts). *Let $a < b$ be real numbers. Let $f, g\colon [a,b] \to \mathbb{R}$ be differentiable functions such that f' and g' are Riemann integrable. Then*

$$\int_a^b fg' = f(b)g(b) - f(a)g(a) - \int_a^b f'g$$

Proof. Since f is differentiable on $[a,b]$ it is continuous on $[a,b]$ by Proposition 1.5. So, f is Riemann integrable by Theorem 3.6, and then fg' is Riemann integrable by Theorem 4.4. Similarly, $f'g$ is Riemann integrable.

Since f, g are differentiable, Theorem 1.6(iv) says that fg is differentiable and $(fg)' = f'g + fg'$. Since $f'g$ and fg' are Riemann integrable, $f'g + fg'$ is Riemann integrable by Theorem 3.3(i). So, applying Theorem 4.1,

$$\int_a^b (f'g + g'f) = \int_a^b (fg)' = f(b)g(b) - f(a)g(a).$$

□

Theorem 4.6 (Change of Variables, version 1). *Let $a < b$ be real numbers. Let $\phi\colon [a,b] \to [\phi(a), \phi(b)]$ be a differentiable function such that $\phi(a) < \phi(b)$ and such that ϕ' is Riemann integrable. Let $f\colon [\phi(a), \phi(b)] \to \mathbb{R}$ be continuous on $[\phi(a), \phi(b)]$. Then $(f \circ \phi)\phi'$ is Riemann integrable on $[a,b]$, and*

$$\int_a^b (f \circ \phi)\phi' = \int_{\phi(a)}^{\phi(b)} f.$$

Proof. Since ϕ is differentiable, ϕ is continuous. Then $f \circ \phi$ is continuous, since it is the composition of two continuous functions. For $t \in [\phi(a), \phi(b)]$, define $F(t) := \int_{\phi(a)}^{t} f$. Recall that f is Riemann integrable by Theorem 3.6. Now, $F'(t) = f(t)$ for all $t \in [\phi(a), \phi(b)]$ by the second fundamental theorem of calculus, Theorem 4.2. For any $x \in [a,b]$, define $g(x) := F \circ \phi(x)$. Then, by the Chain Rule, we have

$$g'(x) = F'(\phi(x))\phi'(x) = f(\phi(x))\phi'(x).$$

Note that g' is the product of two Riemann integrable functions, so g' is Riemann integrable (from Theorem 4.4). So, applying the first fundamental theorem of calculus, Theorem 4.1, we get

$$\int_a^b (f \circ \phi)\phi' = \int_a^b g' = g(b) - g(a) = F(\phi(b)) - F(\phi(a)) = F(\phi(b)) = \int_{\phi(a)}^{\phi(b)} f.$$

□

The following theorem is more difficult to prove, but it allows a change of variables for any Riemann integrable function f.

Theorem 4.7 (**Change of Variables, version 2**). *Let $a < b$ be real numbers. Let $\phi\colon [a,b] \to [\phi(a),\phi(b)]$ be differentiable, strictly monotone increasing function. Assume that ϕ' is Riemann integrable on $[a,b]$. Let $f\colon [\phi(a),\phi(b)] \to \mathbb{R}$ be Riemann integrable on $[\phi(a),\phi(b)]$. Then $(f\circ\phi)\phi'$ is Riemann integrable on $[a,b]$, and*

$$\int_a^b (f\circ\phi)\phi' = \int_{\phi(a)}^{\phi(b)} f.$$

5. Appendix: Notation

Let A, B be sets in a space X. Let m, n be a nonnegative integers.

$$\begin{aligned}
\mathbb{Z} &:= \{\ldots,-3,-2,-1,0,1,2,3,\ldots\}, \text{ the integers}\\
\mathbb{N} &:= \{0,1,2,3,4,5,\ldots\}, \text{ the natural numbers}\\
\mathbb{Z}_+ &:= \{1,2,3,4,\ldots\}, \text{ the positive integers}\\
\mathbb{Q} &:= \{m/n\colon m,n\in\mathbb{Z}, n\neq 0\}, \text{ the rationals}\\
\mathbb{R} &\text{ denotes the set of real numbers}\\
\mathbb{R}^* &= \mathbb{R}\cup\{-\infty\}\cup\{+\infty\} \text{ denotes the set of extended real numbers}\\
\mathbb{C} &:= \{x+y\sqrt{-1} : x,y\in\mathbb{R}\}, \text{ the complex numbers}\\
\emptyset &\text{ denotes the empty set, the set consisting of zero elements}\\
\in &\text{ means ``is an element of.'' For example, } 2\in\mathbb{Z} \text{ is read as ``2 is an element of } \mathbb{Z}\text{.''}\\
\forall &\text{ means ``for all''}\\
\exists &\text{ means ``there exists''}\\
\mathbb{F}^n &:= \{(x_1,\ldots,x_n)\colon x_i\in\mathbb{F},\ \forall\, i\in\{1,\ldots,n\}\}\\
A\subseteq B &\text{ means } \forall\, a\in A, \text{ we have } a\in B, \text{ so } A \text{ is contained in } B\\
A\setminus B &:= \{x\in A\colon x\notin B\}\\
A^c &:= X\setminus A, \text{ the complement of } A\\
A\cap B &\text{ denotes the intersection of } A \text{ and } B\\
A\cup B &\text{ denotes the union of } A \text{ and } B
\end{aligned}$$

Let E be a subset of $\mathbb{R}\cup\{-\infty\}\cup\{+\infty\}$. Let $(a_n)_{n=0}^{\infty}$ be a sequence of real numbers.

$$\begin{aligned}
\sup(E) &\text{ denotes the smallest upper bound of } E\\
\inf(E) &\text{ denotes the largest lower bound of } E\\
\limsup(a_n)_{n=0}^{\infty} &:= \lim_{n\to\infty}\sup_{m\geq n}(a_n)_{n=m}^{\infty}\\
\liminf(a_n)_{n=0}^{\infty} &:= \lim_{n\to\infty}\inf_{m\geq n}(a_n)_{n=m}^{\infty}
\end{aligned}$$

5.1. **Set Theory.** Let X, Y be sets, and let $f: X \to Y$ be a function. The function $f: X \to Y$ is said to be **injective** (or **one-to-one**) if and only if: for every $x, x' \in V$, if $f(x) = f(x')$, then $x = x'$.

The function $f: X \to Y$ is said to be **surjective** (or **onto**) if and only if: for every $y \in Y$, there exists $x \in X$ such that $f(x) = y$.

The function $f: X \to Y$ is said to be **bijective** (or a **one-to-one correspondence**) if and only if: for every $y \in Y$, there exists exactly one $x \in X$ such that $f(x) = y$. A function $f: X \to Y$ is bijective if and only if it is both injective and surjective.

Two sets X, Y are said to have the same **cardinality** if and only if there exists a bijection from X onto Y.

UCLA Department of Mathematics, Los Angeles, CA 90095-1555
E-mail address: `heilman@math.ucla.edu`