Please provide complete and well-written solutions to the following exercises.

Due October 30, in the discussion section.

Assignment 4

Exercise 1. Let $(a_n)_{n=0}^{\infty}$ be a sequence of real numbers. Then $(a_n)_{n=0}^{\infty}$ is convergent if and only if $(a_n)_{n=0}^{\infty}$ is a Cauchy sequence. (Hint: Given a Cauchy sequence $(a_n)_{n=0}^{\infty}$, use that the rationals are dense in the real numbers to replace each real a_n by some rational a_n', so that $|a_n - a_n'|$ is small. Then, ensure that the sequence $(a_n')_{n=0}^{\infty}$ is a Cauchy sequence of rationals and that $(a_n')_{n=0}^{\infty}$ defines a real number which is the limit of the original sequence $(a_n)_{n=0}^{\infty}$.)

Exercise 2. Let $(a_n)_{n=0}^{\infty}$, $(b_n)_{n=0}^{\infty}$ be real convergent sequences. Let x, y be real numbers such that $x = \lim_{n\to\infty} a_n$, $y = \lim_{n\to\infty} b_n$.

(i) The sequence $(a_n + b_n)_{n=0}^{\infty}$ converges to $x + y$. That is,
$$\lim_{n\to\infty}(a_n + b_n) = (\lim_{n\to\infty} a_n) + (\lim_{n\to\infty} b_n).$$

(ii) The sequence $(a_n b_n)_{n=0}^{\infty}$ converges to xy. That is,
$$\lim_{n\to\infty}(a_n b_n) = (\lim_{n\to\infty} a_n)(\lim_{n\to\infty} b_n).$$

(iii) For any real number c, the sequence $(ca_n)_{n=0}^{\infty}$ converges to cx. That is,
$$c\lim_{n\to\infty} a_n = \lim_{n\to\infty}(ca_n).$$

(iv) The sequence $(a_n - b_n)_{n=0}^{\infty}$ converges to $x - y$. That is,
$$\lim_{n\to\infty}(a_n - b_n) = (\lim_{n\to\infty} a_n) - (\lim_{n\to\infty} b_n).$$

(v) Suppose $x \neq 0$ and there exists m such that $a_n \neq 0$ for all $n \geq m$. Then $(a_n^{-1})_{n=m}^{\infty}$ converges to x^{-1}. That is,
$$\lim_{n\to\infty} a_n^{-1} = (\lim_{n\to\infty} a_n)^{-1}.$$

(vi) Suppose $x \neq 0$ and there exists m such that $a_n \neq 0$ for all $n \geq m$. Then $(b_n/a_n)_{n=m}^{\infty}$ converges to y/x. That is,
$$\lim_{n\to\infty}(b_n/a_n) = (\lim_{n\to\infty} b_n)/(\lim_{n\to\infty} a_n).$$

(vii) Suppose $a_n \geq b_n$ for all $n \geq 0$. Then $x \geq y$.

(Hint: you can save time by using some of these statements to prove the others. For example: (iii) follows from (ii); (iv) follows from (i); and (vi) follows from (v) and (ii).)

Exercise 3. For each natural number n, let a_n be a real number such that $|a_n| \leq 2^{-n}$. Define $b_n := a_1 + a_2 + \cdots + a_n$. Prove that the sequence $(b_n)_{n=0}^{\infty}$ is convergent.

Exercise 4. Let E be a subset of $\mathbf{R}^*$. Then the following statements hold.

- For every $x \in E$, we have $x \leq \sup(E)$ and $x \geq \inf(E)$.
- Let $M \in \mathbf{R}^*$ be an upper bound for E, so that $x \leq M$ for all $x \in E$. Then $\sup(E) \leq M$.
- Let $M \in \mathbf{R}^*$ be a lower bound for E, so that $x \geq M$ for all $x \in E$. Then $\inf(E) \geq M$.

(Hint: it may be helpful to break into cases concerning whether or not E contains $+\infty$ or $-\infty$.)

Exercise 5. Let $(a_n)_{n=m}^{\infty}$ be a sequence of real numbers. Let x be the extended real number $x := \sup(a_n)_{n=m}^{\infty}$. Then $a_n \leq x$ for all $n \geq m$. Also, for any $M \in \mathbf{R}^*$ which is an upper bound for $(a_n)_{n=m}^{\infty}$ (so that $a_n \leq M$ for all $n \geq m$), we have $x \leq M$. Finally, for any $y \in \mathbf{R}^*$ such that $y < x$, there exists at least one integer n with $n \geq m$ such that $y < a_n \leq x$. (Hint: use the previous exercise.)

Exercise 6. Let $(a_n)_{n=m}^{\infty}$ be a bounded sequence of real numbers. Assume also that $(a_n)_{n=m}^{\infty}$ is monotone increasing. That is, $a_{n+1} \geq a_n$ for all $n \geq m$. Then the sequence $(a_n)_{n=m}^{\infty}$ is convergent. In fact,

$$\lim_{n \to \infty} a_n = \sup(a_n)_{n=m}^{\infty}.$$

(Hint: use the previous exercise.)

Exercise 7. Let $(a_n)_{n=m}^{\infty}$ be a sequence of real numbers that converges to a real number x. Then x is a limit point of $(a_n)_{n=m}^{\infty}$. Moreover, x is the only limit point of $(a_n)_{n=m}^{\infty}$.

Exercise 8. Let $(a_n)_{n=m}^{\infty}$ be a sequence of real numbers. Let L^+ be the limit superior of this sequence, and let L^- be the limit inferior of this sequence. (Note that $L^+, L^- \in \mathbf{R}^*$.)

(iii) $\inf(a_n)_{n=m}^{\infty} \leq L^- \leq L^+ \leq \sup(a_n)_{n=m}^{\infty}$.
(iv) If c is any limit point of $(a_n)_{n=m}^{\infty}$, then $L^- \leq c \leq L^+$.
(v) If L^+ is finite, then it is a limit point of $(a_n)_{n=m}^{\infty}$. If L^- is finite, then it is a limit point of $(a_n)_{n=m}^{\infty}$.
(vi) Let c be a real number. If $(a_n)_{n=m}^{\infty}$ converges to c, then $L^+ = L^- = c$. Conversely, if $L^+ = L^- = c$, then $(a_n)_{n=m}^{\infty}$ converges to c.