Please provide complete and well-written solutions to the following exercises.

Due December 4, in the discussion section.

Assignment 8

Exercise 1. Let X be a subset of $\mathbf{R}$ and let $f\colon X \to \mathbf{R}$ be a function. Then the following two statements are equivalent.

- f is uniformly continuous on X.
- For any two equivalent sequences $(a_n)_{n=m}^{\infty}$, $(b_n)_{n=m}^{\infty}$, the sequences $(f(a_n))_{n=m}^{\infty}$, $(f(b_n))_{n=m}^{\infty}$ are also equivalent sequences.

Exercise 2. Give an example of a continuous function $f\colon \mathbf{R} \to (0,\infty)$ such that, for any real number $0 < \varepsilon < 1$, there exists $x \in \mathbf{R}$ such that $f(x) = \varepsilon$.

Exercise 3. Let X be a subset of $\mathbf{R}$ and let $f\colon X \to \mathbf{R}$ be a uniformly continuous function. Let x_0 be an adherent point of X. Then $\lim_{x\to x_0} f(x)$ exists (and so it is a real number.)

Exercise 4. Let X be a subset of $\mathbf{R}$, and let $f\colon X \to \mathbf{R}$ be a uniformly continuous function. Assume that E is a bounded subset of X. Then $f(E)$ is also bounded.

Exercise 5. Let X be a subset of $\mathbf{R}$, let x_0 be a limit point of X, and let $f\colon X \to \mathbf{R}$ be a function. If f is differentiable at x_0, then f is also continuous at x_0.

Exercise 6. Let X be a subset of $\mathbf{R}$, let x_0 be a limit point of X, let $f\colon X \to \mathbf{R}$ be a function, and let L be a real number. Then the following two statements are equivalent.

- f is differentiable at x_0 on X with derivative L.
- For every $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that, if $x \in X$ satisfies $|x - x_0| < \delta$, then
$$|f(x) - [f(x_0) + L(x - x_0)]| \leq \varepsilon\,|x - x_0|\,.$$

Exercise 7. Let X be a subset of $\mathbf{R}$, let x_0 be a limit point of X, and let $f\colon X \to \mathbf{R}$ and $g\colon X \to \mathbf{R}$ be functions.

(i) If f is constant, so that there exists $c \in \mathbf{R}$ such that $f(x) = c$, then f is differentiable at x_0 and $f'(x_0) = 0$.
(ii) If f is the identity function, so that $f(x) = x$, then f is differentiable at x_0 and $f'(x_0) = 1$.
(iii) If f, g are differentiable at x_0, then $f + g$ is differentiable at x_0, and $(f+g)'(x_0) = f'(x_0) + g'(x_0)$. (**Sum Rule**)
(iv) If f, g are differentiable at x_0, then fg is differentiable at x_0, and $(fg)'(x_0) = f'(x_0)g(x_0) + g'(x_0)f(x_0)$. (**Product Rule**)
(v) If f is differentiable at x_0, and if $c \in \mathbf{R}$, then cf is differentiable at x_0, and $(cf)'(x_0) = cf'(x_0)$.

(vi) If f, g are differentiable at x_0, then $f - g$ is differentiable at x_0, and $(f-g)'(x_0) = f'(x_0) - g'(x_0)$.

(vii) If g is differentiable at x_0, and if $g(x) \neq 0$ for all $x \in X$, then $1/g$ is differentiable at x_0, and $(1/g)'(x_0) = -\frac{g'(x_0)}{(g(x_0))^2}$.

(viii) If f, g are differentiable at x_0, and if $g(x) \neq 0$ for all $x \in X$, then f/g is differentiable at x_0, and

$$(f/g)'(x_0) = \frac{g(x_0)f'(x_0) - f(x_0)g'(x_0)}{(g(x_0))^2}. \qquad \textbf{(Quotient Rule)}$$

Hint: For the product rule, you may need the following identity

$$f(x)g(x) - f(x_0)g(x_0) = f(x)(g(x) - g(x_0)) + g(x_0)(f(x) - f(x_0)).$$

Exercise 8. Let X, Y be subsets of $\mathbf{R}$, let $x_0 \in X$ be a limit point of X, and let $y_0 \in Y$ be a limit point of Y. Let $f\colon X \to Y$ be a function such that $f(x_0) = y_0$ and such that f is differentiable at x_0. Let $g\colon Y \to \mathbf{R}$ be a function that is differentiable at y_0. Then the function $g \circ f\colon X \to \mathbf{R}$ is differentiable at x_0, and

$$(g \circ f)'(x_0) = g'(y_0)f'(x_0).$$

(Hint: recall that it suffices to consider a sequence $(a_n)_{n=0}^{\infty}$ of elements of X converging to x_0. Also, from Exercise 5, f is continuous, so $(f(a_n))_{n=0}^{\infty}$ converges to $f(x_0)$.)

Exercise 9. Let $a < b$ be real numbers, and let $f\colon (a,b) \to \mathbf{R}$ be a function. If $x_0 \in (a,b)$, if f is differentiable at x_0, and if f attains a local maximum or minimum at x_0, then $f'(x_0) = 0$.

Exercise 10. Let $a < b$ be real numbers, and let $f\colon [a,b] \to \mathbf{R}$ be a continuous function which is differentiable on (a,b). Assume that $f(a) = f(b)$. Then there exists $x \in (a,b)$ such that $f'(x) = 0$. (Hint: use Exercise 9 and the Maximum Principle.)

Exercise 11. Let X be a subset of $\mathbf{R}$, let x_0 be a limit point of X, and let $f\colon X \to \mathbf{R}$ be a function. If f is monotone increasing and if f is differentiable at x_0, then $f'(x_0) \geq 0$. If f is monotone decreasing and if f is differentiable at x_0, then $f'(x_0) \leq 0$.

Exercise 12. Let $a < b$ be real numbers, and let $f\colon [a,b] \to \mathbf{R}$ be a differentiable function. If $f'(x) > 0$ for all $x \in [a,b]$, then f is strictly monotone increasing. If $f'(x) < 0$ for all $x \in [a,b]$, then f is strictly monotone decreasing. If $f'(x) = 0$ for all $x \in [a,b]$, then f is a constant function.