Please provide complete and well-written solutions to the following exercises.

Due October 24, at the beginning of class.

Assignment 7

Exercise 1. Compute the Taylor polynomials T_2 and T_3 for the function $f(x) = \dfrac{1}{1+x^2}$ at $a = 1$.

Exercise 2. Compute the Taylor polynomials T_2 and T_3 for the function $f(x) = \dfrac{1}{1+x}$ at $a = 2$. Then, compute the Taylor polynomials T_2 and T_3 for the function f at $a = 1$.

Exercise 3. Let T_n be the nth Taylor polynomial of the function $f(x) = \cos(x)$ at $a = 0$. Find n such that the following error bounds holds:

$$|\cos(.3) - T_n(.3)| \leq 10^{-7}.$$

(It would probably help to use a calculator here.)

Exercise 4. Let T_n be the nth Taylor polynomial of the function $f(x) = \sin(x)$ at $a = 0$. Find n such that the following error bounds holds:

$$|\sin(.5) - T_n(.5)| \leq 2^{-53}.$$

(It would probably help to use a calculator here.)

Exercise 5. Show that, for any integer $n \geq 1$,

$$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}.$$

(Hint: use induction.)

Exercise 6. For the following sequences, say whether or not they converge or diverge as $n \to \infty$. If they converge, find the corresponding limit. If they diverge, determine if they diverge to infinity.

- $a_n = (-1/2)^n$
- $a_n = n!$.
- $a_n = (-1)^n$.
- $a_n = 1 - \frac{1}{n}$.

Exercise 7. Let $a_1 = 1$. For any $n \geq 1$, define

$$a_{n+1} = \sqrt{1 + a_n}.$$

Show that the sequence $a_1, a_2, \ldots$ is increasing and it is contained in the interval $[1, 2]$. (Hint: use induction to show that $1 \leq a_n \leq 2$ for all $n \geq 1$. Then use induction to show that the sequence is increasing.) (Second hint: to show that the sequence is increasing, the inductive hypothesis is that $a_n \leq a_{n+1}$. Assuming this hypothesis, you should then deduce $a_{n+1} \leq a_{n+2}$.)

Since the sequence is increasing and bounded, $\lim_{n\to\infty} a_n$ exists. So, this exercise gives a meaning to the infinite repeated radical

$$\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}}$$

(Optional: compute $\lim_{n\to\infty} a_n$.) (Computing this limit will not be covered on the quiz.)

Exercise 8. Let M be a positive real number. Recall that the Babylonian square root algorithm is a sequence defined as follows. If $M > 1$ we define $a_1 = M$. If $0 < M \leq 1$, we define $a_1 = 1$. Then, if n is a positive integer and we know a_n, we compute a_{n+1} by

$$a_{n+1} = a_n - \frac{a_n^2 - M}{2a_n} = \frac{1}{2}\left(a_n + \frac{M}{a_n}\right).$$

Show that $\{a_n\}$ is a decreasing sequence. Show also that $\{a_n\}$ is a positive sequence. Conclude that the sequence $\{a_n\}$ converges to some real number L. (In fact, a_n converges to $\sqrt{M}$, but you do not have to show this.)