MATH 126 HOMEWORK SOLUTIONS

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Contents

1. Homework 1

Exercise 1.1. Let $f(x)=e^{x^2}$. Find the equation of the tangent line to f at $x_0=1$.

Solution. We have $f'(x)=2xe^{x^2}$, so the tangent line is $y=f(1)+(x-1)f'(1)=e+(x-1)2e$. □

Exercise 1.2. Differentiate $f(x)=e^{(x^2+3x+2)^2}$.

Solution. From the chain rule, $f'(x)=2(x^2+3x+2)(2x+3)e^{(x^2+3x+2)^2}$. □

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Exercise 1.3. Let $f(x) = x^2e^x$. Find the critical points of f. Classify these critical points as local maxima, local minima, or neither.

Solution. We have $f'(x) = x^2e^x + 2xe^x = x(x+2)e^x$. Since $e^x > 0$ for all x, we have $f'(x) = 0$ if and only if $x = 0$ or $x = -2$. Since $f'(x) > 0$ for $x < -2$ and $f'(x) < 0$ for $-2 < x < 0$, we know that $x = -2$ is a local maximum. Since $f'(x) < 0$ for $-2 < x < 0$ and $f'(x) > 0$ for $x > 0$, we know that $x = 0$ is a local minimum. □

Exercise 1.4. Let j be a positive integer. Let $f(x) = x^je^x$. Find the critical points of f. Classify these critical points as local maxima, local minima, or neither.

Solution. We have $f'(x) = x^je^x + jx^{j-1}e^x = x^{j-1}(x+j)e^x$. Since $e^x > 0$ for all x, we have $f'(x) = 0$ if and only if $x = 0$ or $x = -j$. We first consider the case that j is even. Since $f'(x) > 0$ for $x < -j$ and $f'(x) < 0$ for $-j < x < 0$, we know that $x = -j$ is a local maximum. Since $f'(x) < 0$ for $-j < x < 0$ and $f'(x) > 0$ for $x > 0$, we know that $x = 0$ is a local minimum. We now consider the case that j is odd. Since $f'(x) < 0$ for $x < -j$ and $f'(x) > 0$ for $-j < x < 0$, we know that $x = -j$ is a local minimum. Since $f'(x) > 0$ for $-j < x < 0$ and $f'(x) > 0$ for $x > 0$, we know that $x = 0$ is neither a local minimum nor a local maximum. □

Exercise 1.5. Evaluate the integral

$$\int_0^5 e^{-10y}dy.$$

Solution. $\int_0^5 e^{-10y}dy = (-1/10)e^{-10y}|_{y=0}^{y=5} = (-1/10)(e^{-50} - 1) = (1/10)(1 - e^{-50})$. □

Exercise 1.6. Evaluate the integral

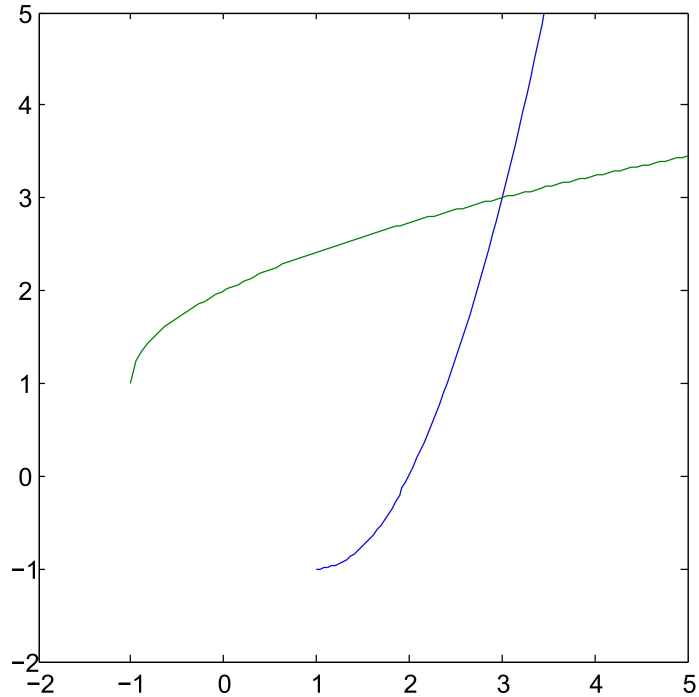
$$\int_1^3 ye^{3y^2}dy.$$

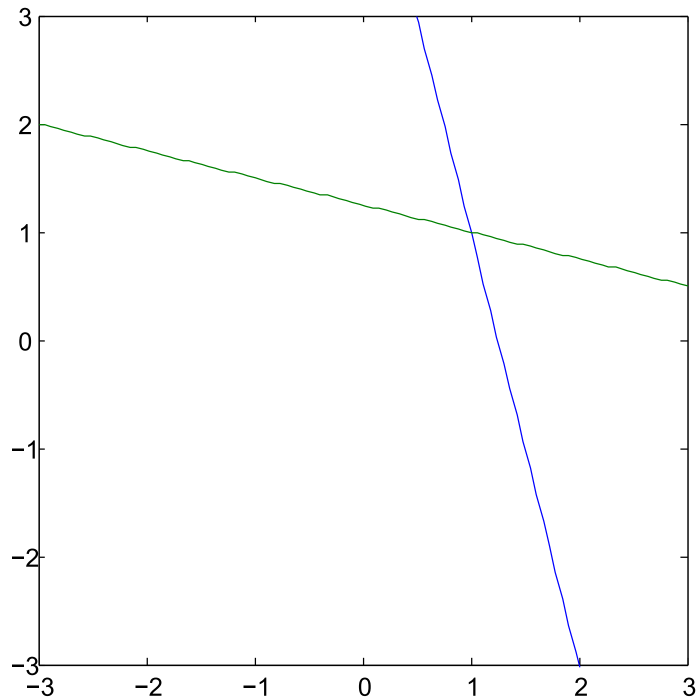
Solution. Substituting $u = 3y^2$ so that $du = 6ydy$, we have $\int_1^3 ye^{3y^2}dy = \int_{u=3}^{u=27}(1/6)e^u du = (1/6)(e^{27} - e^3)$. □

Exercise 1.7. Let $f(x) = x^2 - 2x$, where f has the domain $x \geq 1$. Find a formula for f^{-1}, and then plot both f and f^{-1}.

Solution. Note that $f'(x) = 2x - 2$ and $f'(x) \geq 0$ when $x \geq 1$. So, f has range $[f(1), \infty) = [-1, \infty)$. So, we need a function g with domain $[-1, \infty)$ such that $f(g(x)) = x$. That is, g must satisfy $(g(x))^2 - 2g(x) = x$. That is, $(g(x) - 1)^2 - 1 = x$, so $(g(x) - 1)^2 = x + 1$, for all $x \geq -1$. If $x \geq -1$, then $x + 1 \geq 0$. So, we must have $g(x) - 1 = \sqrt{x+1}$. That is, $g(x) = 1 + \sqrt{x+1}$, where $x \geq -1$. □

Exercise 1.8. Let $f(x) = 5 - 4x$. Graph f and f^{-1} together. Evaluate $f'(x)$ at $x = 1/2$, evaluate $(f^{-1})'(x)$ at $x = f(1/2)$, and verify that $1/f'(1/2) = (f^{-1})'(f(1/2))$.





Solution. To find the inverse, we need $g(x)$ such that $f(g(x)) = x$. That is, g must satisfy $5 - 4g(x) = x$. That is, $g(x) = (1/4)(5 - x)$. Note that $f'(1/2) = -4$, $f(1/2) = 3$, and $g'(f(1/2)) = g'(3) = -1/4$. So, $1/f'(1/2) = 1/(-4) = -1/4 = g'(f(1/2))$.

□

Exercise 1.9. Show that the function $f(x) = (1-x)^3$ has an inverse on the domain $(-\infty, \infty)$. Find a formula for the derivative of this inverse.

Solution. Note that f has range $(-\infty, \infty)$. So, we try to find a function g with domain $(-\infty, \infty)$ and range $(-\infty, \infty)$ such that $f(g(x)) = x$, that is so that $(1 - g(x))^3 = x$. That is, g should satisfy $g(x) = 1 - x^{1/3}$. Indeed, we have $f(g(x)) = (1-(1-x^{1/3}))^3 = (x^{1/3})^3 = x$, and $g(f(x)) = 1 - ((1-x)^3)^{1/3} = 1 - (1-x) = x$, so g is the inverse of g. □

Exercise 1.10. Let $f(x) = \sqrt{x^2 + 6x}$, where f has the domain $x \leq -6$. Let g be the inverse of f. Compute $g'(4)$.

Solution. We first solve for $\sqrt{x^2 + 6x} = 4$. We get $x^2 + 6x = 16$, so that $(x+8)(x-2) = 0$, so that $x = 2$ or $x = -8$. Since $x \leq -6$, we must choose $x = -8$. Then $f(-8) = 4$, so $g(4) = g(f(-8)) = -8$. So, $g'(4) = 1/f'(g(4))$. Since $f'(x) = (1/2)(x^2 + 6x)^{-1/2}(2x + 6)$, we have $f'(g(4)) = f'(-8) = (-5)(64 - 48)^{-1/2} = -5/4$, so $g'(4) = -4/5$. □

2. Homework 2

Exercise 2.1. Find

$$\lim_{x\to\infty} \frac{e^{e^x}}{e^x}.$$

Solution. $\lim_{x\to\infty} \frac{e^{e^x}}{e^x} \overset{L'H}{=} \lim_{x\to\infty} \frac{e^x e^{e^x}}{e^x} = \lim_{x\to\infty} e^{e^x} = \infty$. □

Exercise 2.2. Using the Pythagorean Theorem, derive the following formula.

$$\cos(\sin^{-1}(x)) = \sqrt{1 - x^2}.$$

Solution. The right triangle with angle $\sin^{-1}(x)$ and hypotenuse 1 has height x. So, this triangle has base $\sqrt{1-x^2}$ by the Pythagorean theorem. So, $\cos(\sin^{-1}(x)) = \sqrt{1-x^2}/1 = \sqrt{1-x^2}$. □

Exercise 2.3 (The shape of hanging cables and chains). Suppose we have a cable hanging between two poles of equal height. We will derive the shape of the hanging cable. That is, we will find a function $y = f(x)$, with $x = 0$ the midpoint of the cable, such that the cable follows the curve $y = f(x)$. At the outset, we assume that the function $y = f(x)$ is differentiable.

Consider a segment of the cable from $x = 0$ to $x = b > 0$. We consider this segment of cable as a single body. Then there are three distinct forces acting on this segment of cable. At $x = 0$, we assume that the curve $y = f(x)$ has a horizontal tangent. First, there is a tension force T_0 pulling the cable in the negative x direction at $x = 0$, so that this force is tangent to the curve at $x = 0$. Second, there is a tension force T pulling the cable at $x = b$, and this force is also tangent to the curve at $x = b$. Third, the force of gravity of the segment of chain from $x = 0$ to $x = b$ pulls straight down. This third force is denoted $-\rho g s(b)$, where g is the force of gravity, ρ is a constant, and $s(b)$ is the length of the chain from $x = 0$ to $x = b$.

Adding all three forces together, we must get zero, since the cable is hanging in equilibrium. Suppose at $x = b$ that the tangent line to $y = f(x)$ makes an angle θ with the x-axis. Then the sum of forces in the x-direction is $-T_0 + T\cos(\theta)$, and the sum of forces in the y-direction is $-\rho g s(b) + T\sin(\theta)$. So, $T_0 = T\cos(\theta)$ and $\rho g s(b) = T\sin(\theta)$. Since $(df/dx)(b) = \sin(\theta)/\cos(\theta)$, we have

$$\frac{df}{dx}(b) = \frac{\rho g s(b)}{T_0}. \qquad (*)$$

Now, $s(b)$ is the length of $f(x)$ from $x = 0$ to $x = b$. Let $h > 0$, and assume that s is differentiable. From the linear approximation of the derivative, we have $f(b+h) \approx f(b) + hf'(b)$. Consider the right triangle with vertices $(b, f(b))$, $(b+h, f(b+h))$, and $(b+h, f(b))$. The length of the hypotenuse of this triangle is approximately $s(b+h) - s(b)$. Also, from the Pythagorean Theorem, the length of the hypoteuse of this triangle is $\sqrt{h^2 + (f(b+h) - f(b))^2}$. Combining these facts,

$$s(b+h) - s(b) \approx \sqrt{h^2 + (f(b+h) - f(b))^2} \approx \sqrt{h^2 + h^2(f'(b))^2}.$$

So, dividing both sides by $h > 0$ we have $(s(b+h) - s(b))/h \approx \sqrt{1 + (f'(b))^2}$. Letting $h \to 0^+$, and then taking a derivative of $(*)$, we have derived the following equation

$$\frac{d^2 f}{dx^2}(b) = \frac{\rho g}{T_0}\sqrt{1 + \left(\frac{df}{dx}(b)\right)^2}. \qquad (**)$$

We can now solve for f. Show that the shape of the cable is described by

$$y = f(x) = \frac{T_0}{\rho g}\cosh\left(\frac{\rho g x}{T_0}\right).$$

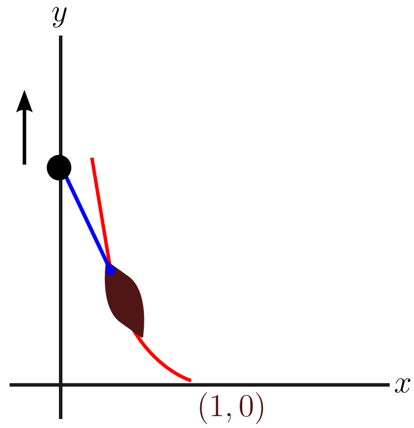
The shape of the hanging cable, known as a catenary curve, also appears in architecture, such as the St. Louis Gateway Arch or Gaudi's Sagrada Familia. The idea is to freeze the hanging cable in its position, and then flip it upside down to produce an arch. Then we can repeat the derivation above to see that the forces of the arch are all the same as in the case of the chain (though the signs of the forces are flipped). Also, for a very small segment of the arch, we can essentially neglect the gravitational force exerted on this segment. So, by reviewing the above analysis, the force on any particular point in the arch will be directed along the arch itself. Therefore, the arch is very stable.

Solution. Recall that $\cosh'(x) = \sinh(x)$, $\sinh'(x) = \cosh(x)$, $1 + \sinh^2(x) = \cosh^2(x)$, and $\cosh(x) > 0$ for all x, so

$$\sqrt{1 + (f'(x))^2} = \sqrt{1 + \cosh^2(\rho g x/T_0)} = \sinh(\rho g x/T_0) = f''(x)T_0/(\rho g).$$

□

Exercise 2.4 (**Towing an unconstrained object**)**.** Suppose I am standing on the shore of a straight river, and I am pulling on a rope of length 1 connected to the front of a canoe. I am walking at a constant speed in the positive y-direction. Suppose the canoe is in the river, and the canoe's front is at a distance x from the river shore. Suppose the initial position of the canoe is $(1, 0)$. As I move in the positive y-direction, the canoe's front is pulled along a curve denoted by $y = f(x)$. If the rope is taut, it will always be tangent to the curve $f(x)$.



Consider the right triangle formed by: me on the shore, the point $(x, f(x))$ and the y-axis. Then the height of this triangle in the y-direction is $\sqrt{1-x^2}$. Therefore,

$$\frac{df}{dx} = -\frac{\sqrt{1-x^2}}{x}$$

Show that $f(x) = \mathrm{sech}^{-1}(x) - \sqrt{1-x^2}$ satisfies $\frac{df}{dx} = -\frac{\sqrt{1-x^2}}{x}$.

Solution. Recall that $(d/dx)\mathrm{sech}^{-1}(x) = -1/(x\sqrt{1-x^2})$, so $f'(x) = -\frac{1}{x\sqrt{1-x^2}} + \frac{x}{\sqrt{1-x^2}} = \frac{-1+x^2}{x\sqrt{1-x^2}} = -\frac{\sqrt{1-x^2}}{x}$, using $0 < x < 1$. □

Exercise 2.5. In this exercise, all velocities are measured with respect to meters per second, and $c \approx 3 \times 10^8$ meters per second is the speed of light.

Einstein's special theory of relativity implies the following fact. Suppose that I am running with velocity v_1, where we consider the earth to be fixed. Suppose I then throw a baseball at a velocity v_2, relative to my own frame of reference. Then, relative to the earth, the baseball does **not** travel with velocity $v_1 + v_2$. Relative to the earth, the baseball travels with velocity V, where

$$\tanh^{-1}(V/c) = \tanh^{-1}(v_1/c) + \tanh^{-1}(v_2/c).$$

Using a calculator or other electronic aid, find the velocity V of the baseball relative to the earth if $v_1 = 2 \times 10^8$ and $v_2 = 2.5 \times 10^8$.

Solution. We have $\tanh^{-1}(V/c) = \tanh^{-1}(2/3) + \tanh^{-1}(5/6)$, so $V \approx c\tanh(.8047 + 1.1989) \approx c \cdot .9643 \approx 2.89 \times 10^8$ meters per second. □

Exercise 2.6. Compute the following limits

$$\lim_{x\to -5} \frac{x^2 - 25}{5 - 4x - x^2}.$$

$$\lim_{x\to\infty} \frac{9x - 4}{4 - 2x}.$$

$$\lim_{x\to 0} \frac{\sin(3x)}{\sin(5x)}.$$

Solution. $\lim_{x\to -5} \frac{x^2-25}{5-4x-x^2} \overset{L'H}{=} \lim_{x\to -5} \frac{2x}{-4-2x} = \frac{-10}{-4+10} = -5/3$.

$\lim_{x\to\infty} \frac{9x-4}{4-2x} \overset{L'H}{=} -9/2$.

$\lim_{x\to 0} \frac{\sin(3x)}{\sin(5x)} \overset{L'H}{=} \lim_{x\to 0} \frac{3\cos(3x)}{5\cos(3x)} = 3/5$.

□

3. Homework 3

Exercise 3.1. Compute

$$\lim_{x\to\pi/2} (x-\pi/2)\tan(x).$$

Solution.

$$\lim_{x\to\pi/2} (x-\pi/2)\tan(x) = \lim_{x\to\pi/2} \frac{x-\pi/2}{1/\tan(x)}$$
$$\overset{L'H}{=} \lim_{x\to\pi/2} \frac{1}{-(\cos(x))^{-2}/\tan^2(x)} = \lim_{x\to\pi/2} -\sin^2(x) = -1.$$

□

Exercise 3.2. What is

$$\lim_{x\to\sqrt{3}} \frac{\tan^{-1}(x)-\pi/3}{x-\sqrt{3}}?$$

Solution. Recall that $\tan(\pi/3) = \sqrt{3}$, and $\tan^{-1}(\sqrt{3}) = \pi/3$, so $\lim_{x\to\sqrt{3}} \frac{\tan^{-1}(x)-\pi/3}{x-\sqrt{3}} \overset{L'H}{=} \lim_{x\to\sqrt{3}} \frac{1}{1+x^2} = 1/4$. □

Exercise 3.3. Using integration by parts, compute the following integrals

- $\int t^2 e^{4t} dt$.
- $\int x\sin(x/2) dx$.
- $\int_0^{1/\sqrt{2}} 2x\sin^{-1}(x^2) dx$.
- $\int_2^4 x(\ln x)^2 dx$.

Solution.

$$\int t^2 e^{4t} dt = \int t^2 (1/4)(d/dt)e^{4t} dt = (1/4)t^2e^{4t} - (1/2)\int te^{4t} dt$$
$$= (1/4)t^2e^{4t} - (1/8)\int t(d/dt)e^{4t} dt = (1/4)t^2e^{4t} - (1/8)te^{4t} + (1/8)\int e^{4t} dt$$
$$= (1/4)t^2e^{4t} - (1/8)te^{4t} + (1/32)e^{4t} + C.$$

$$\begin{aligned}\int x\sin(x/2)dx &= \int 2x(d/dx)(-\cos(x/2))dx = -2x\cos(x/2) + \int 2\cos(x/2)dx\\ &= -2x\cos(x/2) + 4\sin(x/2) + C.\end{aligned}$$

Recall that $(d/dt)\sin^{-1}(t) = 1/\sqrt{1-t^2}$, so substituting $u = x^2$, then substituting $v = 1-u^2$,

$$\begin{aligned}\int_0^{1/\sqrt{2}} 2x\sin^{-1}(x^2)dx &= \int_0^{1/2}\sin^{-1}(u)du = \int_0^{1/2}\sin^{-1}(u)(d/du)udu\\ &= [u\sin^{-1}(u)]_0^{1/2} - \int_0^{1/2}\frac{u}{\sqrt{1-u^2}}du\\ &= (1/2)\sin^{-1}(1/2) + (1/2)\int_1^{3/4} v^{-1/2}dv = (1/2)\sin^{-1}(1/2) + [v^{1/2}]_1^{3/4}\\ &= (1/2)(\pi/6) + (\sqrt{3}/2) - 1 = (\pi/12) + (\sqrt{3}/2) - 1\end{aligned}$$

$$\begin{aligned}\int_2^4 x(\ln(x))^2dx &= \int_2^4 (\ln x)^2(d/dx)(x^2/2)dx = [(\ln x)^2(x^2/2)]_2^4 - \int_2^4 x\ln(x)dx\\ &= 8(\ln 4)^2 - 2(\ln(2))^2 - \int_2^4 \ln(x)(d/dx)(x^2/2)dx\\ &= 8(\ln 4)^2 - 2(\ln(2))^2 - [(x^2/2)\ln(x)]_2^4 + \int_2^4 (x/2)dx\\ &= 8(\ln 4)^2 - 2(\ln(2))^2 - 8\ln(4) + 2\ln(2) + (1/4)(16-4).\end{aligned}$$

□

Exercise 3.4. Compute the following integrals

- $\int \sin^4(x)\cos^2(x)dx$.
- $\int \cos(x)\sin^{111}(x)dx$.
- $\int \tan^3(x)\sec^2(x)dx$.

Solution.

$$\int \sin^4(x)\cos^2(x)dx = \int (1-\cos^2(x))^2\cos^2(x)dx = \int \cos^6(x) - 2\cos^4(x) + \cos^2(x)dx$$
$$= \int (1/8)(\cos(2x)+1)^3 - (1/2)(\cos(2x)+1)^2 + (1/2)(\cos(2x)+1)dx$$
$$= \int (1/8)(\cos^3(2x) + 3\cos^2(2x) + 3\cos(2x) + 1)$$
$$- (1/2)(\cos^2(2x) + 2\cos(2x) + 1) + (1/2)(\cos(2x)+1)dx$$
$$= \int (1/8)\cos^3(2x) + (3/8 - 1/2)\cos^2(2x) + (3/8 - 1 + 1/2)\cos(2x) + 1/8 - 1/2 + 1/2dx$$
$$= \int (1/8)(1-\sin^2(2x))\cos(2x) - (1/16)(1+\cos(4x)) - (1/8)\cos(2x) + 1/8dx$$
$$= (1/16)\int (1-u^2)du - (1/16)(x + (1/4)\sin(4x)) - (1/16)\sin(2x) + x/8$$
$$= (1/16)(\sin(2x) - (1/3)\sin^3(2x)) - (1/16)(x + (1/4)\sin(4x)) - (1/16)\sin(2x) + x/8 + C.$$

We substituted $u = \sin(2x)$, so that $du = 2\cos(2x)dx$.

We substitute $u = \sin(x)$ so that $du = \cos(x)dx$,

$$\int \cos(x)\sin^{111}(x)dx = \int u^{111}du = (1/112)u^{112} = (1/112)\sin^{112}(x) + C.$$

We substitute $u = \tan(x)$, so that $du = \sec^2(x)dx$, and

$$\int \tan^3(x)\sec^2(x)dx = \int u^3du = (1/4)u^4 = (1/4)\tan^4(x) + C.$$

□

4. Homework 4

Exercise 4.1. Using trigonometric substitution, compute

$$\int \frac{x^2}{\sqrt{9-x^2}}\,dx.$$

Solution. We substitute $x = 3\sin\theta$, so that $dx = 3\cos\theta d\theta$, and $\sqrt{9-x^2} = 3\sqrt{1-\sin^2\theta} = |\cos\theta|$, $du = \cos\theta$. Assuming $-3 \le x \le 3$, we have $-\pi/2 \le \theta \le \pi/2$, so $\cos\theta \ge 0$ and

$|\cos\theta| = \cos\theta$, so

$$\int \frac{x^2}{\sqrt{9-x^2}}\,dx = \int 9\sin^2\theta\cos\theta\,|\cos\theta|^{-1}\,d\theta \int 9\sin^2\theta d\theta$$
$$= \int (9/2)(1-\cos(2\theta))d\theta = (9/2)(\theta - (1/2)\sin(2\theta))$$
$$= (9/2)\sin^{-1}(x/3) - (9/4)\sin(2\sin^{-1}(x/3))$$
$$= (9/2)\sin^{-1}(x/3) - (9/2)\sin(\sin^{-1}(x/3))\cos(\sin^{-1}(x/3))$$
$$= (9/2)\sin^{-1}(x/3) - (9/2)(x/3)\sqrt{1-x^2/9}$$
$$= (9/2)\sin^{-1}(x/3) - (x/2)\sqrt{9-x^2} + C.$$

□

Exercise 4.2. Using trigonometric substitution, compute

$$\int \frac{dx}{\sqrt{25x^2-4}}.$$

Solution. Substituting $x = (2/5)(1/\cos\theta)$, so that $dx = (2/5)\sin\theta/(\cos^2\theta)d\theta$, and $\sqrt{25x^2-4} = \sqrt{4(\cos\theta)^{-2}-4} = 2\,|\tan\theta|$. Assuming $|\tan\theta| = \tan\theta$, we get

$$\int \frac{dx}{\sqrt{25x^2-4}} = \int (1/5)\sin\theta(\cos\theta)^{-2}\,|\tan\theta|^{-1}\,d\theta$$
$$= \int (1/5)(\cos\theta)^{-1}d\theta = (1/5)\ln|\sec\theta + \tan\theta|$$
$$= (1/5)\ln\left|5x/2 + \sqrt{(25/4)x^2-1}\right| = (1/5)\ln\left|5x/2 + \sqrt{(25/4)x^2-1}\right| + C.$$

We used $\sec\theta = 5x/2$, so $\cos(\theta) = 2/(5x)$, so $\sin\theta = \sin(\cos^{-1}(2/(5x))) = \sqrt{1-4/(25x^2)}$, so $\tan\theta = (5x/2)\sqrt{1-4/(25x^2)} = \sqrt{(25/4)x^2-1}$, assuming $x \geq 0$. □

Exercise 4.3. Evaluate the following integrals using the method of partial fractions.

$$\int \frac{dx}{x^2+2x}.$$

$$\int_{1/2}^{1} \frac{y+4}{y^2+y}dy.$$

$$\int \frac{4x^2-21x}{(x-3)^2(2x+3)}dx.$$

Solution. We have $1/(x^2+2x) = A/x + B/(x+2)$, so $1 = A(x+2) + Bx$, so setting $x = 0$ gives $A = 1/2$, and setting $x = -2$ gives $B = -1/2$. So,

$$\int \frac{dx}{x^2+2x} = \int \frac{1}{2x} - \frac{1}{2(x+2)}dx = (1/2)\ln|x| - (1/2)\ln|x+2| + C.$$

We have $(y+4)/(y^2+y) = A/y + B/(y+1)$, so $y+4 = A(y+1)+By$, so setting $y=0$ gives $A=4$, and setting $y=-1$ gives $B=-3$. So,

$$\int_{1/2}^{1} \frac{y+4}{y^2+y}dy = \int_{1/2}^{1} \frac{4}{y} - \frac{3}{y+1}dy = [4\ln|y| - 3\ln|y+1|]_{y=1/2}^{y=1}$$
$$= -3\ln(2) - 4\ln(1/2) + 3\ln(3/2) = \ln(16\cdot 27/8^2) = \ln(27/4).$$

We have

$$\frac{4x^2-21x}{(x-3)^2(2x+3)} = \frac{A}{(x-3)} + \frac{B}{(x-3)^2} + \frac{C}{2x+3}.$$

So,

$$4x^2 - 21x = A(x-3)(2x+3) + B(2x+3) + C(x-3)^2.$$

Setting $x=3$, we have $36-63=-27=B(9)$, so $B=-3$. Setting $x=-3/2$, we have $C(9/2)^2 = 9+63/2$, so $C=2$. Equating the x^2 terms on both sides of our equality, we have $4 = 2A+C$, so that $A=(4-C)/2=1$. Therefore,

$$\int \frac{4x^2-21x}{(x-3)^2(2x+3)}dx = \ln|x-3| + 3(x-3)^{-1} + \ln|2x+3| + C'.$$

□

Exercise 4.4. Compute the following integral

$$\int \frac{xdx}{(x^2-1)^{3/2}}.$$

Solution. Substituting $u = x^2-1$ so that $du = 2xdx$, we have $\int \frac{xdx}{(x^2-1)^{3/2}} = (1/2)\int u^{-3/2}du = -u^{-1/2} = -(x^2-1)^{-1/2}$. □

Exercise 4.5. Compute the following integral:

$$\int \ln(x^4-1)dx.$$

Solution. Integrating by parts,

$$\int \ln(x^4-1)dx = \int \ln(x^4-1)(d/dx)xdx = x\ln(x^4-1) - \int \frac{x(4x^3)}{x^4-1}dx.$$

Now, write $4x^4 = 4(x^4-1)+4$, so that $4x^4/(x^4-1) = 4 + 4/(x^4-1)$. Then

$$\int \ln(x^4-1)dx = x\ln(x^4-1) - 4x - \int \frac{4dx}{x^4-1}.$$

Now, we have $1/(x^4-1) = A/(x+1) + B/(x-1) + (Cx+D)/(x^2+1)$, so that

$$1 = A(x-1)(x^2+1) + B(x+1)(x^2+1) + (Cx+D)(x^2-1).$$

Setting $x=-1$, we get $1 = A(-2)(2)$, so $A=-1/4$. Setting $x=1$, we get $1=B(2)(2)$, so $B=1/4$. Equating the x^3 terms on both sides of our equality, we have $0=A+B+C$, so

$C = -A - B = 1/4 - 1/4 = 0$. Equating the x^2 terms on both sides of our equality, we have $0 = -A + B + D$, so $D = A - B = -1/4 - 1/4 = -1/2$. In conclusion,

$$\int \ln(x^4 - 1)dx = x\ln(x^4 - 1) - 4x + \ln|x+1| - \ln|x-1| + 2\tan^{-1}(x) + C'.$$

□

5. Homework 5

Exercise 5.1. Evaluate

$$\int_0^\infty xe^{-x}dx.$$

Solution. Integrating by parts for any $0 < b < \infty$, we have $\int_0^b xe^{-x}dx = \int_0^b x(-d/dx)e^{-x}dx = [-xe^{-x}]_0^b + \int_0^b e^{-x}dx = -be^{-b} - e^{-b} + 1$. Letting $b \to \infty$, we have $\lim_{b\to\infty} be^{-b} = 0$ and $\lim_{b\to\infty} e^{-b} = 0$, so $\int_0^\infty xe^{-x}dx = \lim_{b\to\infty}\int_0^b xe^{-x}dx = 1$. □

Exercise 5.2. Compute

$$\int_{-1}^1 \sqrt{|x|}\,dx.$$

Solution. $\int_{-1}^1 \sqrt{|x|}\,dx = \int_0^1 x^{1/2}dx + \int_{-1}^0 (-x)^{1/2}dx = \int_0^1 x^{1/2}dx - \int_1^0 x^{1/2}dx = 2\int_0^1 x^{1/2}dx = 2(2/3) = 4/3$, using the substitution $u = -x$. □

Exercise 5.3. Compute

$$\int_{-1}^2 \frac{1}{x^2}dx.$$

Solution. This integral diverges since $\int_0^2 \frac{1}{x^2}dx = \lim_{t\to 0^+}\int_t^2 x^{-2}dx = \lim_{t\to 0^+}(-x^{-1})_t^2 = \lim_{t\to 0^+}(-2 + t^{-1}) = \infty$. □

Exercise 5.4. Compute the following integral, or show that the integral diverges.

$$\int_0^3 \frac{dx}{(3-x)^{3/2}}.$$

Solution. This integral diverges. We have $\int_0^3 \frac{dx}{(3-x)^{3/2}} = \lim_{t\to 3^-}\int_0^t \frac{dx}{(3-x)^{3/2}} = \lim_{t\to 3^-}(1/2)(3-x)^{-1/2}|_0^t = \lim_{t\to 3^-}(1/2)(3-t)^{-1/2} - (1/2)3^{-1/2} = \infty$. □

Exercise 5.5. Compute the following integral

$$\int_3^5 (9 - x^2)dx.$$

Then, approximate this integral by computing the Trapezoid rule T_N, the Midpoint rule M_N, and Simpson's rule S_N for $N = 4$. Compute also the error bounds for these three integral approximations. Which approximation is the best?

Solution. $\int_3^5 (9 - x^2)dx = [9x - (1/3)x^3]_3^5 = 9(5-3) - (1/3)(125-27) = 18 - (1/3)(98) = -44/3$.

The Trapezoid rule for $N = 4$ uses the points $3, 3.5, 4, 4.5, 5$, so that

$$T_4 = \frac{5-3}{4}((9-3^2)/2 + (9-3.5^2) + (9-4^2) + (9-4.5^2) + (9-5^2)/2) = -59/4 = -14.75.$$

$$M_4 = \frac{5-3}{4}((9-3.25^2) + (9-3.75^2) + (9-4.25^2) + (9-4.75^2)) = -117/8 = -14.625.$$

$$S_4 = \frac{5-3}{12}((9-3^2)+4(9-3.5^2)+2(9-4^2)+4(9-4.5^2)+(9-5^2)) = -44/3 = \int_3^5 (9-x^2)dx.$$

Simpson's rule gives the best approximation, since it actually gives an equality.

Consider $f(x) = 9 - x^2$. Then $f''(x) = -2$ for all x, so we can use $K = 2$ in the error bounds for T_4 and M_4. Using $K = 2$, we observe that the error bounds are correct, since

$$\left| T_4 - \int_3^5 (9-x^2)dx \right| = |-14.75 + 44/3| = 1/12 \leq \frac{2(5-3)^3}{12(4)^2} = 1/12.$$

$$\left| M_4 - \int_3^5 (9-x^2)dx \right| = |-14.625 + 44/3| = 1/24 \leq \frac{2(5-3)^3}{24(4)^2} = 1/24.$$

Finally, since $f^{(4)}(x) = 0$, we can use $K = 0$ in the error bound for S_4. And indeed, this is okay, since

$$\left| S_4 - \int_3^5 (9-x^2)dx \right| = 0.$$

□

Exercise 5.6. Compute the following integral

$$\int_0^4 x^3 dx.$$

Then, approximate this integral by computing the Trapezoid rule T_N, the Midpoint rule M_N, and Simpson's rule S_N for $N = 4$. Compute also the error bounds for these three integral approximations. Which approximation is the best?

Solution. $\int_0^4 x^3 dx = [(1/4)x^4]_0^4 = 64$.

The Trapezoid rule for $N = 4$ uses the points $0, 1, 2, 3, 4$, so that

$$T_4 = \frac{4}{4}(0^3/2 + 1^3 + 2^3 + 3^3 + 4^3/2) = 68.$$

$$M_4 = \frac{4}{4}((1/2)^3 + (3/2)^3 + (5/2)^3 + (7/2)^3) = 62.$$

$$S_4 = \frac{4}{12}(0^3/2 + 4 \cdot 1^3 + 2 \cdot 2^3 + 4 \cdot 3^3 + 4^3) = 64$$

Simpson's rule gives the best approximation, since it actually gives an equality.

Consider $f(x) = x^3$. Then $f''(x) = 6x$ for all x. So, $|f''(x)| = 6|x| \leq 24$ when $0 \leq x \leq 4$. So we can use $K = 24$ in the error bounds for T_4 and M_4. Using $K = 24$, we observe that the error bounds are correct, since

$$\left|T_4 - \int_0^4 x^3 dx\right| = |68 - 64| = 4 \leq \frac{24(4-0)^3}{12(4)^2} = 8.$$

$$\left|M_4 - \int_0^4 x^3 dx\right| = |62 - 64| = 2 \leq \frac{24(4-0)^3}{24(4)^2} = 4.$$

Finally, since $f^{(4)}(x) = 0$, we can use $K = 0$ in the error bound for S_4. And indeed, this is okay, since

$$\left|S_4 - \int_0^4 x^3 dx\right| = 0.$$

□

Exercise 5.7. Compute the surface area of revolution about the x-axis over the interval $[0, 1]$ of the function

$$y = 2x + 1.$$

Solution. The surface area is given by $\int_0^1 2\pi y(x)\sqrt{1 + (y'(x))^2} = \int_0^1 2\pi(2x + 1)\sqrt{5}dx = 2\pi\sqrt{5}\int_0^1 (2x + 1)dx = 2\pi\sqrt{5}(2) = 4\pi\sqrt{5}$. □

Exercise 5.8. Compute the surface area of revolution about the x-axis over the interval $[1, 2]$ of the function

$$y = \sqrt{9 - x^2}$$

Solution. The surface area is given by

$$\int_1^2 2\pi y(x)\sqrt{1 + (y'(x))^2}dx = 2\pi \int_1^2 \sqrt{9 - x^2}\sqrt{1 + (\frac{-x}{\sqrt{9 - x^2}})^2}dx$$
$$= 2\pi \int_1^2 \sqrt{9 - x^2}\sqrt{\frac{9}{9 - x^2}}dx = 2\pi \int_1^2 3dx = 18\pi.$$

□

6. Homework 6

Exercise 6.1. Let $r, h > 0$ be constants. Let $f(x) = \frac{r}{h}x$, where $0 \leq x \leq h$. Compute the volume by revolution of f over $0 \leq x \leq h$, where f is rotated around the x-axis. You should find a formula for the volume of a circular cone of radius r and height h. (Hint: use the disk method.)

Solution.

$$\int_0^h \pi(f(x))^2 dx = \pi r^2 h^{-2} \int_0^h x^2 dx = \pi r^2 h^{-2}[(1/3)x^3]_{x=0}^h = (1/3)\pi r^2 h.$$

□

Exercise 6.2. Compute the volume of the solid obtained by revolving the region enclosed by the ellipse $x^2 + 9y^2 = 9$ about the x-axis.

Solution. Since $y^2 = 1 - x^2/9$, we have $y = \sqrt{1 - x^2/9}$ or $y = -\sqrt{1 - x^2/9}$. Either curve, when rotated around the x-axis, produces the same solid region, which has volume

$$\pi \int_{-3}^{3} (f(x))^2 dx = \pi \int_{-3}^{3} (1 - x^2/9) dx = \pi(6 - x^3/27)_{x=-3}^{x=3} = \pi(6 - 1 - 1) = 4\pi.$$

□

Exercise 6.3. Consider the region in the plane contained between the lines $y = x$, $y = 0$, $x = 0$ and $x = 1$. Compute the volume obtained by rotating this region around the y-axis.

Solution. The cylinder of height 1 and radius 1 has volume π. A circular cone with radius 1 and height 1 has volume $\pi \int_0^1 y^2 dy = \pi/3$ by the disk method. So, the volume of the stated region is the difference of these numbers, i.e. $\pi - (\pi/3) = 2\pi/3$. Alternatively, if we use cylindrical shells with $f(x) = x$, we get

$$2\pi \int_0^1 x^2 dx = 2\pi/3.$$

□

Exercise 6.4. Suppose we take the region between the curves $x = y^2 - 3y$ and $x = 2y - y^2$ and we revolve this region around the line $x = 3$. Write an integral that computes the volume of the resulting solid region. You do NOT have to evaluate the integral.

Solution. We first find the intersection of the curves. Intersection occurs when $y^2 - 3y = 2y - y^2$, i.e. when $2y^2 - 5y = 0$, i.e. $y(2y - 5) = 0$. Also, when $y = 1$, we have $y^2 - 3y = -2$ and $2y - y^2 = -1$, so $2y - y^2 > y^2 - 3y$ when $0 < y < 5/2$. So, intersection occurs when $y = 0$ and when $y = 5/2$. Also, when $0 < y < 5/2$, the function $f(y) = y^2 - 3y = y(y - 3)$ has maximum value 0, and the function $g(y) = 2y - y^2 = y(2 - y)$ has maximum value 0. That is, both curves lie on the left side of the y-axis. And the curve f lies to the left of the curve g. So, using the disk method, the volume is given by

$$\pi \int_0^{5/2} (y^2 - 3y - 3)^2 - (2y - y^2 - 3)^2 dy.$$

□

Exercise 6.5. Find the arc length of $y = 2^{-4}x^4 + (1/2)x^{-2}$ over the interval $[1, 4]$. (Hint: write $1 + (y')^2$ as the square of something.)

Solution. Note that $y'(x) = (1/4)x^3 - x^{-3}$, so $1 + (y')^2 = 1 + (1/16)x^6 + x^{-6} - (1/2) = (1/2) + (1/16)x^6 + x^{-6} = ((1/4)x^3 - x^{-3})^2$. The arc length is then given by $\int_1^4 \sqrt{1 + (y'(x))^2} dx = \int_1^4 (1/2)(x^3 - x^{-3}) dx = (1/2)[(1/4)x^4 + (1/2)x^{-2}]_1^4 = (1/2)[64 - 1/4 + (1/32) - 1/2]$. □

Exercise 6.6. Using a comparison of integrals, show that the arc length of $y = x^{4/3}$ over $[1, 2]$ is greater than or equal to $5/3$.

Solution. Note that $y'(x) = (4/3)x^{1/3}$, so $1 + (y'(x))^2 = 1 + (16/9)x^{2/3}$. If $x \geq 1$, then $1 + (y'(x))^2 \geq 1 + 16/9 = 25/9$. So, the arc length is equal to $\int_1^2 \sqrt{1 + (y'(x))^2}dx \geq \int_1^2 \sqrt{25/9}dx = 5/3$. □

Exercise 6.7. Compute the force on one side of a square plate of side length 2 meters, where the plate is submerged vertically in a tank of water, and one side of the square is tangent the surface of the water.

Solution. The force is given by $\rho g \int_0^2 y2dy = 4\rho g$. □

Exercise 6.8. Compute the force on one side of a circular plate of radius 3 meters, where the plate is submerged vertically in a tank of water, and the top of the circle is tangent to the surface of the water.

Solution. The force is given by $\rho g \int_0^6 y2\sqrt{9 - (y-3)^2}dy$. Changing variables so that $u = y - 3$, we have $\int_0^6 y2\sqrt{9 - (y-3)^2}dy = \int_{-3}^3 (u+3)2\sqrt{9 - u^2}du$. We know that $\int_{-3}^3 \sqrt{9 - u^2}du = (9/2)\pi$, since it is half the area of the disc of radius 3. (Alternatively, we could use trigonometric substitution $u = 3\sin\theta$.) Also, substituting $v = 9 - u^2$ so that $dv = -2udu$, we have $\int_{-3}^3 u\sqrt{9 - u^2}du = 0$. In conclusion, the force is equal to $3(2)\rho g(9/2)\pi = \rho g 27\pi$. □

7. Homework 7

Exercise 7.1. Compute the Taylor polynomials T_2 and T_3 for the function $f(x) = \dfrac{1}{1+x^2}$ at $x = 1$.

Solution. We have

$$f'(x) = -2x/(1+x^2)^2.$$

$$\begin{aligned} f''(x) &= [(1+x^2)^2(-2) + 2x(2)(1+x^2)(2x)]/(1+x^2)^4 \\ &= [(1+x^2)(-2) + 8x^2]/(1+x^2)^3 = (6x^2-2)/(1+x^2)^3. \end{aligned}$$

$$f'''(x) = [(1+x^2)^3(12x) - (6x^2-2)3(1+x^2)^2(2x)]/(1+x^2)^4.$$

So, $f(1) = 1/2$, $f'(1) = -2/(2^2) = -1/2$, $f''(1) = 4/8 = 1/2$, and $f'''(1) = [8(12) - 4(3)(4)(2)]/16 = 0$. So,

$$T_2(x) = 1/2 - (1/2)(x-1) + (1/4)(x-1)^2, \qquad T_3(x) = T_2(x).$$

□

Exercise 7.2. Compute the Taylor polynomials T_2 and T_3 for the function $f(x) = \dfrac{1}{1+x}$ at $x = 2$. Then, compute the Taylor polynomials T_2 and T_3 for the function f at $x = 1$.

Solution. We have

$$f'(x) = -1/(1+x)^2.$$

$$f''(x) = 2(1+x)/(1+x)^4 = 2/(1+x)^3.$$

$$f'''(x) = -6(1+x)^2/(1+x)^6 = -6/(1+x)^4.$$

So, $f(2) = 1/3$, $f'(2) = -1/9$, $f''(2) = 2/27$, $f'''(2) = -6/81 = -2/27$, and at $x = 2$, we have

$$T_2(x) = (1/3) - (1/9)(x-2) + (1/27)(x-2)^2.$$
$$T_3(x) = (1/3) - (1/9)(x-2) + (1/27)(x-2)^2 - (1/81)(x-2)^3$$

Also, $f(1) = 1/2$, $f'(1) = -1/4$, $f''(1) = 2/8 = 1/4$, and $f'''(1) = -6/16 = -3/8$. So, at $x = 1$, we have

$$T_2(x) = 1/2 - (1/4)(x-1) + (1/8)(x-1)^2.$$
$$T_3(x) = 1/2 - (1/4)(x-1) + (1/8)(x-1)^2 - (1/16)(x-1)^3.$$

□

Exercise 7.3. Let T_n be the nth Taylor expansion of the function $f(x) = \cos(x)$ at $x = 0$. Find n such that the following error bounds holds:

$$|\cos(.3) - T_n(.3)| \leq 10^{-7}.$$

Solution. Note that all derivatives of cosine are bounded by 1. So, the error formula for the Taylor expansion says we need the following error bound to hold: $(1/3)^{n+1}/(n+1)! < 10^{-7}$. So, $n = 6$ suffices, since then $(1/3)^{n+1}/(n+1)! = 3^{-7}/5040 \approx 9.07 \times 10^{-8}$. □

Exercise 7.4. Let T_n be the nth Taylor expansion of the function $f(x) = \sin(x)$ at $x = 0$. Find n such that the following error bounds holds:

$$|\sin(.5) - T_n(.5)| \leq 2^{-53}.$$

Solution. Note that all derivatives of sine are bounded by 1. So, the error formula for the Taylor expansion says we need the following error bound to hold: $(1/2)^{n+1}/(n+1)! < 2^{-53}$. So, $n = 14$ suffices, since then $(1/2)^{n+1}/(n+1)! = 2^{-15}/15! \approx 2.33 \times 10^{-17}$, while $2^{-53} \approx 1.11 \times 10^{-16}$. □

Exercise 7.5. Show that, for any integer $n \geq 1$,

$$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}.$$

(Hint: use induction.)

Solution. We induct on n. The base case is $n = 1$, in which the left side is $1^2 = 1$ and the right side is $1(1+1)(2+1)/6 = 1$. So, the base case holds. We now do the inductive step. We assume that the assertion holds for n, and we prove that it holds for the case $n+1$. Then, using the inductive hypothesis for the first n terms in the sum

$$\sum_{j=1}^{n+1} j^2 = (n+1)^2 + \sum_{j=1}^{n} j^2 (n+1)^2 + \frac{n(n+1)(2n+1)}{6}$$

$$= (n+1)[(n+1) + n(2n+1)/6] = \frac{n+1}{6}(6n + 6 + 2n^2 + n) = \frac{n+1}{6}(2n^2 + 7n + 6) = \frac{n+1}{6}(2n + 3$$

That is, the assertion holds in the case $n+1$. We have completed the inductive step, therefore the proof is complete. □

Exercise 7.6. For the following sequences, say whether or not they converge or diverge as $n \to \infty$. If they converge, find the corresponding limit. If they diverge, determine if they diverge to infinity.

- $a_n = (-1/2)^n$
- $a_n = n!$.
- $a_n = (-1)^n$.
- $a_n = 1 - \frac{1}{n}$.

Solution. The first sequence converges to zero, since $|a_n| = 1/(2^n)$, and $\lim_{n\to\infty} 2^n = \infty$, so $\lim_{n\to\infty} 1/2^n = 0$, so $\lim_{n\to\infty} a_n = 0$ as well.

The second sequence diverges to infinity, since $n!$ becomes an arbitrarily large number as $n \to \infty$. (In particular, $n! \geq n$, so $\lim_{n\to\infty} n! \geq \lim_{n\to\infty} n = \infty$.)

The third sequence diverges, since it does not get close to any particular number as $n \to \infty$. (For $n \geq 1$, $|a_n - a_{n+1}| = 2$, so it cannot occur that the limit as $n \to \infty$ of a_n exists.) Also, $|a_n| = 1$ for any $n \geq 1$, so the sequence does not diverge to infinity, since its absolute value is bounded for all n.

The last sequence converges to 1 since

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (1 - 1/n) = 1 - \lim_{n\to\infty} (1/n) = 1 - 0 = 1.$$

□

Exercise 7.7. Let $a_1 = 1$. For any $n \geq 1$, define

$$a_{n+1} = \sqrt{1 + a_n}.$$

Show that the sequence $a_1, a_2, \ldots$ is increasing and it is contained in the interval $[1, 2]$. (Hint: use induction to show that $1 \leq a_n \leq 2$ for all $n \geq 1$. Then use induction to show that the sequence is increasing.) (Second hint: to show that the sequence is increasing, the inductive hypothesis is that $a_n \leq a_{n+1}$. Assuming this hypothesis, you should then deduce $a_{n+1} \leq a_{n+2}$.)

Since the sequence is increasing and bounded, $\lim_{n\to\infty} a_n$ exists. So, this exercise gives a meaning to the infinite repeated radical

$$\sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \cdots}}}}$$

(Optional: compute $\lim_{n\to\infty} a_n$.) (Computing this limit will not be covered on the quiz.)

Solution. We first show by induction that $1 \leq a_n \leq 2$. The base case $n = 1$ holds since $a_1 = 1$ so $1 \leq a_1 \leq 2$. We now show the inductive step. Assume that $1 \leq a_n \leq 2$. We need to show that $1 \leq a_{n+1} \leq 2$. Since $1 \leq a_n \leq 2$, we have

$$2 \leq 1 + a_n \leq 3 \quad \Rightarrow \quad 1 \leq \sqrt{2} \leq \sqrt{1 + a_n} \leq \sqrt{3} \leq 2 \quad \Rightarrow \quad 1 \leq a_{n+1} \leq 2.$$

The last line used the definition of a_{n+1}. The inductive step is complete. We have therefore shown that $1 \leq a_n \leq 2$ for all $n \geq 1$.

We now show that $a_n \leq a_{n+1}$ for all $n \geq 1$. The base case is $n = 1$. Since $a_1 = 1$ and $a_2 = \sqrt{1+1} = \sqrt{2}$ we have $a_1 \leq a_2$. So, the base case is true. We now do the inductive step. Assume that $a_n \leq a_{n+1}$. We need to show that $a_{n+1} \leq a_{n+2}$. That is, we need to show that

$$\sqrt{1+a_n} \leq \sqrt{1+a_{n+1}}.$$

Using the inductive hypothesis, we have

$$a_n \leq a_{n+1} \quad \Rightarrow \quad 1+a_n \leq 1+a_{n+1} \quad \Rightarrow \quad \sqrt{1+a_n} \leq \sqrt{1+a_{n+1}}.$$

The inductive step is complete. Therefore, $a_n \leq a_{n+1}$ for all $n \geq 1$, i.e. the sequence is increasing.

(Optional:) Since the sequence is increasing and bounded by 2, it converges to some limit L (since monotonic bounded sequences converge). We have the informal expression

$$L = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}}.$$

Since $L = \lim_{n\to\infty} a_n$, we also have

$$L = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{1+a_n} = \sqrt{1+\lim_{n\to\infty} a_n} = \sqrt{1+L}.$$

That is, $L^2 = L+1$, so that $L^2 - L - 1 = 0$. That is,

$$L = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}.$$

Since $1 \leq a_n \leq 2$ for all $n \geq 1$, we must have $1 \leq L \leq 2$. So, L must be equal to the positive root of $L^2 - L - 1 = 0$. That is,

$$L = \frac{1+\sqrt{5}}{2} \approx 1.618034...$$

This number is sometimes called the golden ratio.

□

Exercise 7.8. Let M be a positive real number. Recall that the Babylonian square root algorithm is a sequence defined as follows. If $M > 1$ we define $a_1 = M$. If $0 < M \leq 1$, we define $a_1 = 1$. Then, if n is a positive integer and we know a_n, we compute a_{n+1} by

$$a_{n+1} = a_n - \frac{a_n^2 - M}{2a_n} = \frac{1}{2}\left(a_n + \frac{M}{a_n}\right).$$

Show that $\{a_n\}$ is a decreasing sequence. Show also that $\{a_n\}$ is a positive sequence. Conclude that the sequence $\{a_n\}$ converges to some real number L. (In fact, a_n converges to $\sqrt{M}$, but you do not have to show this.)

Solution. We know that a_1 is positive, and if a_n is positive for any $n \geq 1$, then a_{n+1} is also positive, since it is defined as a sum of positive numbers. It therefore follows by induction that a_n is positive for any $n \geq 1$. We now show that $a_n \geq \sqrt{M}$ by induction.

In the base case, we have $a_1 = M > \sqrt{M}$ (when $M > 1$) and $a_1 = 1 \geq \sqrt{M}$ (when $0 < M \leq 1$), so the base case holds. We now do the inductive step. Assume $a_n \geq \sqrt{M}$. We need to show that $a_{n+1} \geq \sqrt{M}$. That is, we need to show that

$$\frac{1}{2}\left(a_n + \frac{M}{a_n}\right) \geq \sqrt{M}.$$

Consider the function

$$f(x) = \frac{1}{2}(x + M/x).$$

Then $f'(x) = (1/2)[1 - M/x^2]$. That is, $f(\sqrt{M}) = \sqrt{M}$, and f is increasing for $x > \sqrt{M}$. That is, $f(x) > \sqrt{M}$ when $x > \sqrt{M}$. By the inductive hypothesis, $a_n \geq \sqrt{M}$, so $a_{n+1} = f(a_n) \geq \sqrt{M}$ as well. Having finished the inductive step, we conclude that $a_n \geq \sqrt{M}$ for all $n \geq 1$.

We now show that the sequence is decreasing by induction. That is, we show that $a_n \geq a_{n+1}$ for all $n \geq 1$. The base case is $n = 1$. When $M > 1$, $a_1 = M$ and $a_2 = \frac{1}{2}(M + \frac{M}{M}) \leq \frac{1}{2}(M+M) = M$, so $a_1 \geq a_2$ when $M > 1$. When $0 < M \leq 1$, $a_1 = 1$ and $a_2 = \frac{1}{2}(M+\frac{M}{1}) = M$, so $a_1 \geq a_2$ when $0 < M \leq 1$. So, the base case is true. We now do the inductive step. Assume that $a_n \geq a_{n+1}$. We need to show that $a_{n+1} \geq a_{n+2}$. That is, we need to show that

$$\frac{1}{2}\left(a_n + \frac{M}{a_n}\right) \geq \frac{1}{2}\left(a_{n+1} + \frac{M}{a_{n+1}}\right).$$

Using the inductive hypothesis, we have $a_n \geq a_{n+1}$. Also, $a_n \geq a_{n+1} \geq \sqrt{M}$, from above. Also from above, the function $f(x) = \frac{1}{2}(x+M/x)$ is increasing when $x > \sqrt{M}$. Consequently, $f(a_n) \geq f(a_{n+1})$, since $a_n \geq a_{n+1}$. The inductive step is complete. Therefore, $a_n \geq a_{n+1}$ for all $n \geq 1$, i.e. the sequence is decreasing. □

8. Homework 8

Exercise 8.1. For each of the following sequences, determine whether or not the sequence converges. (You may need to use the Squeeze Theorem or monotonicity.) If the sequence converges, compute its limit as $n \to \infty$.

- $a_n = n/2^n$.
- $a_n = 1 + (-1)^n$.
- $a_n = 1/(0.9)^n$.
- $a_n = (\sin n)/n$.
- $a_n = \left(1 - \frac{1}{n}\right)^n$.
- $a_n = (1/n)^{1/(\ln n)}$.

Solution.

- $a_n = n/2^n$. This sequence converges to zero. Note that $n \leq 2 \cdot 2^{n/2}$ for all $n \geq 1$. To see this, note that $1 \leq 2 \cdot 2^{1/2}$, and if $f(x) = x$ and $g(x) = 2 \cdot 2^{x/2}$, then $f(1) \leq g(1)$, and $f'(x) = 1 \leq 2^{x/2} = g'(x)$ for all $x \geq 1$. So, $0 \leq a_n \leq 2 \cdot 2^{n/2}2^{-n} = 2 \cdot 2^{-n/2} \to 0$ as $n \to \infty$. So, by the Squeeze Theorem, $\lim_{n\to\infty} a_n = 0$.

- $a_n = 1 + (-1)^n$. This sequence does not converge. If n is even, then $a_n = 2$, and if n is odd, then $a_n = 0$. So, a_n does not converge.
- $a_n = 1/(0.9)^n$. This sequence does not converge. We have $a_n = 1/(9/10)^n = (10/9)^n \to \infty$ as $n \to \infty$.
- $a_n = (\sin n)/n$. This sequence converges to zero. Since $-1 \leq \sin(x) \leq 1$ for all x, we have $-1/n \leq (\sin n)/n \leq 1/n$. So, by the Squeeze Theorem, since $1/n \to 0$ and $-1/n \to 0$ as $n \to \infty$, we must have $a_n \to 0$ as $n \to \infty$.
- $a_n = \left(1 - \frac{1}{n}\right)^n$. We defined $e^x = \lim_{n\to\infty}(1 + x/n)^n$. So, this sequence converges to e^{-1}.
- $a_n = (1/n)^{1/(\ln n)}$. Let $f(x) = (1/x)^{1/(\ln(x))}$. Then $\ln f(x) = \frac{1}{\ln(x)}\ln(1/x) = -\frac{\ln x}{\ln x} = -1$. So, $\ln f(n) = -1$ for all $n > 1$. So, $f(n) = e^{-1}$ for all n. So, this (constant) sequence converges to e^{-1} as $n \to \infty$.

□

Exercise 8.2. From class, we saw that the sequence $a_n = (-1)^n$ does not converge as $n \to \infty$. Similarly, the sequence $b_n = (-1)^{n+1}$ does not converge as $n \to \infty$. However, $a_n + b_n = 0$ for all n, so $a_n + b_n$ does converge as $n \to \infty$. Therefore, in this case, $\lim_{n\to\infty}(a_n + b_n)$ is not equal to $\lim_{n\to\infty} a_n$ plus $\lim_{n\to\infty} b_n$. Explain how this does not contradict the theorem (limit laws for sequences) which stated $\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$.

Solution. We only can conclude that $\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$ when the limit $\lim_{n\to\infty} a_n$ exists, and when the limit $\lim_{n\to\infty} b_n$ exists. In the case $a_n = (-1)^n$ and $b_n = (-1)^{n+1}$, neither of these limits exist. So, it does not follow that $\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$. □

Exercise 8.3. Determine whether or not the following series converge or diverge. If the series converges, find its sum.

- $\sum_{n=0}^{\infty}(\sqrt{2})^n$.
- $\displaystyle\sum_{n=0}^{\infty}\left(\frac{1}{\sqrt{2}}\right)^n$.
- $\displaystyle\sum_{n=0}^{\infty}\frac{\cos(n\pi)}{5^n}$.
- $\displaystyle\sum_{n=1}^{\infty}\frac{n^n}{n!}$.
- $\displaystyle\sum_{n=1}^{\infty}\ln\left(\frac{n}{n+1}\right)$.

Solution.

- $\sum_{n=0}^{\infty}(\sqrt{2})^n$. This series diverges. Note that $(\sqrt{2})^n \to \infty$ as $n \to \infty$. So, the Divergence test implies that the series $\sum_{n=0}^{\infty}(\sqrt{2})^n$ diverges.

- $\sum_{n=0}^{\infty}\left(\frac{1}{\sqrt{2}}\right)^n$. This series converges, since it is a geometric series. The series converges to $1/(1-1/\sqrt{2})=\sqrt{2}/(\sqrt{2}-1)$.
- $\sum_{n=0}^{\infty}\frac{\cos(n\pi)}{5^n}$. This series converges, since it is a geometric series. We can rewrite the series as $\sum_{n=0}^{\infty}\frac{\cos(n\pi)}{5^n}=\sum_{n=0}^{\infty}(-1)^n/5^n=\sum_{n=0}^{\infty}(-1/5)^n=1/(1-(-1/5))=5/6$.
- $\sum_{n=1}^{\infty}\frac{n^n}{n!}$. This series diverges. Since $n^n=n\cdot n\cdots n$ a total of n times, and since $n!=n(n-1)(n-2)\cdots 2\cdot 1$, it always holds that $n^n/n!\geq 1$. So, the Divergence test implies that the series $\sum_{n=1}^{\infty}\frac{n^n}{n!}$ does not converge.
- $\sum_{n=1}^{\infty}\ln\left(\frac{n}{n+1}\right)$. This series diverges. Consider the Nth partial sum

$$S_N=\sum_{n=1}^{N}\ln(n/(n+1))=\sum_{n=1}^{N}(\ln(n)-\ln(n+1)).$$

Then the series is a telescoping series, since $S_N=\ln(1)-\ln(2)+\ln(2)-\ln(3)+\ln(3)-\cdots+\ln(N)-\ln(N+1)$, so that $S_N=-\ln(N+1)$. Since $S_N\to-\infty$ as $N\to\infty$, the partial sums do not converge, so the series does not converge.

□

Exercise 8.4. Suppose a ball is dropped from a height of 4 meters. Each time the ball hits the ground after falling from a height of h meters, the ball rebounds to the height of $(3/4)h$ meters. Find the total distance the ball travels up and down. Then, find the total number of seconds that the ball is moving. (Hint: the formula $s=(4.9)t^2$ implies that $t=\sqrt{|s|/4.9}$.)

Solution. The ball first falls 4 meters, then it goes up $4(3/4)=3$ meters, then down $4(3/4)=3$ meters, then it goes up $4(3/4)^2=9/4$ meters, then down $3(3/4)^2$ meters, then it goes up $3(3/4)^3$ meters, and so on. So, the total distance travelled is $4+8\sum_{n=1}^{\infty}(3/4)^n=4+8\frac{3/4}{1-3/4}=4+8\frac{3/4}{1/4}=4+8(3)=28$ meters. To find the traversal time, note that the ball falls 4 meters from resting position, which lasts $\sqrt{4/4.9}$ seconds. Then, at the next peak of travel, the ball falls 3 meters from resting position, so the time between the first and second hits to the ground is $2\sqrt{3/4.9}$ seconds, followed by $2\sqrt{3(3/4)/4.9}$ seconds, and so on. So, the total travel time is $\sqrt{4/4.9}+2\sqrt{3/4.9}\sum_{n=0}^{\infty}(\sqrt{3/4})^n=\sqrt{4/4.9}+2\sqrt{3/4.9}\frac{1}{1-\sqrt{3/4}}\approx 12.58$ seconds. □

9. Homework 9

Exercise 9.1. Using appropriate convergence tests as necessary, determine whether the following series converge or diverge.

- $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$.
- $\displaystyle\sum_{n=1}^{\infty} \frac{2n+1}{n^2+2n+1}$.
- $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n+1)!}$.
- $\displaystyle\sum_{n=1}^{\infty} \frac{n^{1/n}}{n^2}$.
- $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{n^{5/4}}$.

Solution.

- $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$. This sum converges by the integral test. Let $f(x) = 1/(x(\ln(x))^2)$. Then, using the substitution $u = \ln x$, we get
$$\int_2^\infty f(x)dx = \int_2^\infty (\ln(x))^{-2}dx/x = \int_{\ln 2}^\infty u^{-2}du = \lim_{R\to\infty}\int_{\ln 2}^R u^{-2}du$$
$$= \lim_{R\to\infty}[-u^{-1}]_{u=\ln 2}^{u=R} = \lim_{R\to\infty}[-R^{-1} + (\ln 2)^{-1}] = (\ln 2)^{-1} < \infty$$
- $\displaystyle\sum_{n=1}^{\infty} \frac{2n+1}{n^2+2n+1}$. This sum diverges by comparison. Note that $\frac{2n+1}{n^2+2n+1} \geq 1/n$ for all $n \geq 2$, since $n(2n+1) \geq n^2+2n+1$, i.e. $n^2+n^2+n \geq n^2+n+n+1$. Since, the harmonic series $\sum_{n=1}^\infty 1/n$ diverges, the series $\sum_{n=1}^\infty \frac{2n+1}{n^2+2n+1}$ must also diverge.
- $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n+1)!}$. This series converges. For example, we have $0 \leq 1/(n+1)! \leq 1/n^2$ for all $n \geq 1$. So, by comparison, since $\sum_{n=1}^\infty 1/n^2$ converges, we know that $\sum_{n=1}^\infty \frac{1}{(n+1)!}$ converges.
- $\displaystyle\sum_{n=1}^{\infty} \frac{n^{1/n}}{n^2}$. This series converges. Recall that $L = \lim_{x\to\infty} x^{1/x} = 1$. To see this, note that $\ln L = \lim_{x\to\infty}\frac{\ln x}{x} = \lim_{x\to\infty}(1/x) = 0$ by L'Hopital's Rule, so $\ln L = 0$ and $L = 1$. So, there exists a constant $C > 0$ such that $1 \leq n^{1/n} \leq C$ for all $n \geq 1$. So, by comparison, we have $0 \leq n^{1/n}/n^2 \leq C/n^2$, and so the series converges, since $\sum_{n=1}^\infty C/n^2$ converges.
- $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{n^{5/4}}$. This series converges. Note that
$$\lim_{x\to\infty}\frac{\ln x}{x^{1/8}} = \lim_{x\to\infty}\frac{1/x}{(1/8)x^{-7/8}} = \lim_{x\to\infty} 8x^{-1/8} = 0.$$

So, there exists a constant $C > 0$ such that $\ln(n) \leq Cn^{1/8}$ for all $n > 1$. So, by comparison, we have $0 \leq (\ln n)/n^{5/4} \leq C/n^{9/8}$, and so the series converges, since $\sum_{n=1}^{\infty} 1/n^{9/8}$ converges.

□

Exercise 9.2. Using appropriate convergence tests as necessary, determine whether the following series converge absolutely, converge conditionally, or diverge.

- $\frac{1}{2} - \frac{1}{3} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{2^3} - \frac{1}{3^3} + \cdots$.
- $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{3/2}}$.
- $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$.

Solution.

- $\frac{1}{2} - \frac{1}{3} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{2^3} - \frac{1}{3^3} + \cdots$. This series converges, but the alternating series test does not apply, since the terms are not monotone. However, we have $\left|\frac{1}{2^n} - \frac{1}{3^n}\right| = \left|\frac{3^n - 2^n}{6^n}\right| \leq \frac{3^n}{6^n} = 2^{-n}$. So, by comparison, the even partial sums S_N converge, where S_N is the sum of the first N terms of the series, and N is even. The odd partial sums must then converge to the same value, since $|S_{2N} - S_{2N+1}| < 2^{-N}$.
- $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{3/2}}$. This series converges by the alternating series test.
- $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$. This series converges by comparison. We have $-1/n^2 \leq \sin n/n^2 \leq 1/n^2$ for all $n \geq 1$. And $\sum_{n=1}^{\infty} 1/n^2$ converges.

□

Exercise 9.3. Using the Ratio Test, determine whether the series converges, diverges, or that the test is inconclusive.

- $\sum_{n=1}^{\infty} n! e^{-n}$.
- $\sum_{n=1}^{\infty} \frac{\ln n}{n}$.

Solution.

- $\sum_{n=1}^{\infty} n! e^{-n}$. This series diverges. Let $a_n = n! e^{-n}$. Then $|a_{n+1}/a_n| = |(n+1)/e| \to \infty$ as $n \to \infty$. So, the series $\sum_{n=1}^{\infty} n! e^{-n}$ diverges by the ratio test.

- $\displaystyle\sum_{n=1}^{\infty}\frac{\ln n}{n}$. This series diverges by the comparison test since $(\ln n)/n \geq 1/n$ when $n \geq 3$, but the ratio test is inconclusive. Let $a_n = (\ln n)/n$. Then $|a_{n+1}/a_n| = \left|\frac{\ln(n+1)}{\ln n}\frac{n+1}{n}\right| \to 1$ as $n \to \infty$, so the ratio test is inconclusive. (Note that $\lim_{n\to\infty}\frac{n+1}{n} = 1$ and $\lim_{x\to\infty}\frac{\ln(x+1)}{\ln x} = \lim_{x\to\infty}\frac{1/(x+1)}{1/x} = 1$, both by L'Hopital's Rule.)

$\square$

Exercise 9.4. For the following power series, find the radius of convergence R. Describe the set of all points where the power series converges absolutely, describe the set of all points where the power series converges conditionally, and describe the set of all points where the power series diverges.

- $\displaystyle\sum_{n=0}^{\infty}(x+5)^n$.
- $\displaystyle\sum_{n=0}^{\infty}\frac{x^n}{2^n\sqrt{n}}$.
- $\displaystyle\sum_{n=0}^{\infty}\frac{x^n}{n!}$

Solution.

- $\displaystyle\sum_{n=0}^{\infty}(x+5)^n$. Let x be a real number. Let $a_n = (x+5)^n$. From the ratio test, $|a_{n+1}/a_n| = |x+5|$. So, the series diverges when $|x+5| > 1$, and the series converges absolutely when $|x+5| < 1$. In the case $|x+5| = 1$, we have $x = -4$ or $x = -6$. In the case $x = -4$, the series is $\sum_{n=0}^{\infty} 1$, which diverges. In the case $x = -6$, the series is $\sum_{n=0}^{\infty}(-1)^n$, which diverges by the Divergence Test. So, the radius of convergence is $R = 1$.
- $\displaystyle\sum_{n=0}^{\infty}\frac{x^n}{2^n\sqrt{n}}$. Let x be a real number. Let $a_n = x^n 2^{-n} n^{-1/2}$. From the ratio test, $|a_{n+1}/a_n| = |x|\,2^{-1}(n+1)^{-1/2}n^{1/2}$. Note that $\lim_{n\to\infty} n^{1/2}(n+1)^{-1/2} = (\lim_{n\to\infty} n(n+1)^{-1})^{1/2} = 1^{1/2} = 1$ by L'Hopital's Rule. So, $\lim_{n\to\infty}|a_{n+1}/a_n| = |x|/2$. So, by the ratio test, if $|x| < 2$, then the series converges absolutely, and if $|x| > 2$, then the series diverges. In the remaining cases, we have $x = 2$ or $x = -2$. In the case $x = 2$, the series is $\sum_{n=0}^{\infty}\frac{1}{\sqrt{n}}$, which diverges. In the case $x = -2$, the series is $\sum_{n=0}^{\infty}\frac{(-1)^n}{\sqrt{n}}$, which converges conditionally, by the Alternating Series Test. So, the radius of convergence is $R = 2$.
- $\displaystyle\sum_{n=0}^{\infty}\frac{x^n}{n!}$. Let x be a real number. Let $a_n = x^n/n!$. From the ratio test, $|a_{n+1}/a_n| = |x|/|n+1| \to 0$ as $n \to \infty$. So, from the ratio test, the series converges for all x. So, the radius of convergence is $R = \infty$.

□

Exercise 9.5. For the following series, find the radius of convergence R. Describe the set of all points where the series converges absolutely, describe the set of all points where the series converges conditionally, and describe the set of all points where the series diverges. Finally, find the sum of the series as a function of x.

- $\sum_{n=1}^{\infty} \frac{(x-1)^{2n}}{4n}$.
- $\sum_{n=0}^{\infty} n(n-1)x^{n-2}$.

Solution.

- $\sum_{n=1}^{\infty} \frac{(x-1)^{2n}}{4n}$. Let x be a real number. Let $a_n = (x-1)^{2n}/(4n)$. From the ratio test, $|a_{n+1}/a_n| = |x-1|^2 |n| / |n+1| \to (x-1)^2$ as $n \to \infty$. So, from the ratio test, the series converges absolutely when $|x-1| < 1$, and the series diverges when $|x-1| > 1$. So, the radius of convergence is $R = 1$. The only remaining cases are $x = 0$ and $x = 2$. In the case $x = 0$, the series is $\sum_{n=1}^{\infty} 1/(4n)$, which diverges, since it is $(1/4)$ multiplied by the harmonic series. In the case $x = 2$, the series is $\sum_{n=1}^{\infty} 1/(4n)$, which diverges, since it is $(1/4)$ multiplied by the harmonic series.

  Finally, recall that $-\ln(1-x) = \sum_{n=1}^{\infty} x^n/n$, so
$$\sum_{n=1}^{\infty}(x-1)^{2n}/(4n) = -(1/4)\ln(1-(x-1)^2).$$
- $\sum_{n=0}^{\infty} n(n-1)x^{n-2}$. Let x be a real number. Let $a_n = n(n-1)x^{n-2}$. From the ratio test, $|a_{n+1}/a_n| = \left|\frac{n+1}{n-1}\right| |x| \to |x|$ as $n \to \infty$. So, from the ratio test, the series converges absolutely when $|x| < 1$, and the series diverges when $|x| > 1$. So, the radius of convergence is $R = 1$. The only remaining cases are $x = -1$ and $x = 1$. In the case $x = 1$, the series is $\sum_{n=1}^{\infty} n(n-1)$, which diverges, by the Divergence Test. In the case $x = -1$, the series is $\sum_{n=1}^{\infty} n(n-1)(-1)^n$, which diverges, by the Divergence Test.

  Finally, recall that $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, so differentiating this series term-by-term twice, we see that $\sum_{n=0}^{\infty} n(n-1)x^{n-2} = (d^2/dx^2)(1/(1-x)) = 2/(1-x)^3$.

□

Exercise 9.6. Find the Maclaurin series for the following functions. Find also the radius of convergence.

- $\cosh(x) = \frac{e^x + e^{-x}}{2}$.
- $\frac{1-\cos x}{x}$.

- $e^x \ln(1+x)$.

Solution.

- $\cosh(x) = \dfrac{e^x + e^{-x}}{2}$. Recall that $e^x = \sum_{n=0}^{\infty} x^n/n!$, so $e^{-x} = \sum_{n=0}^{\infty}(-1)^n x^n/n!$, so $\cosh(x) = (e^x + e^{-x})/2 = \sum_{n=0}^{\infty}(1/2)(1+(-1)^n)x^n/n! = \sum_{n=0}^{\infty} x^{2n}/(2n)!$. Let x be a real number. Let $a_n = x^{2n}/(2n)!$. From the ratio test, $|a_{n+1}/a_n| = |x|^2/((2n+2)(2n+1)) \to 0$ as $n \to \infty$. So, from the ratio test, the series converges absolutely for all real x. So, the radius of convergence is $R = \infty$.
- $\dfrac{1-\cos x}{x}$. Recall that $\cos(x) = \sum_{n=0}^{\infty}(-1)^n x^{2n}/(2n)!$, so

$$(1-\cos(x))/x = \sum_{n=1}^{\infty}(-1)^{n-1}x^{2n-1}/(2n)!.$$

  Let x be a real number. Let $a_n = (-1)^{n-1}x^{2n-1}/(2n)!$. From the ratio test, $|a_{n+1}/a_n| = |x|^2/((2n+2)(2n+1)) \to 0$ as $n \to \infty$. So, from the ratio test, the series converges absolutely for all real x. So, the radius of convergence is $R = \infty$. (Note that the series converges for all x, even though the function $(1-\cos x)/x$ is technically undefined at $x = 0$.)
- $e^x \ln(1+x)$. Recall that $e^x = \sum_{n=0}^{\infty} x^n/n!$, and $-\ln(1-x) = \sum_{n=1}^{\infty} x^n/n$, so $\ln(1+x) = -(-\ln(1-(-x))) = \sum_{n=1}^{\infty}(-1)^{n-1}x^n/n$. Define $a_n = 1/n!$, and define $b_n = 0$ for $n = 0$ and $b_n = (-1)^{n-1}/n$ for $n > 0$. Then, define $c_n = \sum_{j=0}^{n} a_j b_{n-j}$. Then from the multiplication formula for power series, the Maclaurin series of $e^x \ln(1+x)$ is $\sum_{n=0}^{\infty} x^n c_n$.

□

10. Homework 10

Exercise 10.1. Show that the following series converges to zero.

$$\pi - \frac{\pi^3}{3!} + \frac{\pi^5}{5!} - \frac{\pi^7}{7!} + \cdots$$

Solution. Recall that $\sin(x) = \sum_{n=0}^{\infty} x^{2n+1}(-1)^n/(2n+1)!$, and from the ratio test, this series converges absolutely for all real x. In particular, $\sin(x)$ converges absolutely for $x = \pi$. Since $\sin(\pi) = 0$, the following series therefore converges to zero: $\sum_{n=0}^{\infty} \pi^{2n+1}(-1)^n/(2n+1)!$. □

Exercise 10.2. Express the following integral as an infinite series for $|x| < 1$

$$\int_0^x \ln(1+t^2)dt.$$

Solution. Recall that $-\ln(1-x) = \sum_{n=1}^{\infty} x^n/n$, and this series converges absolutely for all $|x| < 1$. So, $\ln(1+t^2) = -(-\ln(1-(-t^2))) = \sum_{n=1}^{\infty}(-1)^{n-1}t^{2n}/n$. and this series converges absolutely whenever $|t^2| < 1$, i.e. whenever $|t| < 1$. Integrating the series term by term then maintains the radius of convergence. That is, whenever $|x| < 1$, we have

$$\int_0^x \ln(1+t^2)dt = \sum_{n=1}^{\infty}(-1)^{n-1}n^{-1}\int_0^x t^{2n}dt = \sum_{n=1}^{\infty}(-1)^{n-1}n^{-1}\frac{1}{2n+1}x^{2n+1}.$$

□

Exercise 10.3. The following integral often arises in probability theory, in relation to diffusions, Brownian motion, and so on.

$$F(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt.$$

Using a Taylor series for e^{-t^2}, find a Taylor series for F. Then, find the radius of convergence of this series. Finally, compute $F(1/\sqrt{2})$ to four decimal places of accuracy, and then compute $F(2/\sqrt{2})$ to two decimal places of accuracy. The answers should remind you of the concept of standard deviation. (F is also known as a bell curve, or the error function. Optional: give an estimate for $F(3/\sqrt{2})$.)

Solution. Recall that $e^x = \sum_{n=0}^{\infty} x^n/n!$, so $e^{-t^2} = \sum_{n=0}^{\infty}(-1)^n t^{2n}/n!$. So, integrating term by term, we have

$$F(x) = \frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}(-1)^n(n!)^{-1}\int_0^x t^{2n}dt = \frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}(-1)^n(n!)^{-1}\frac{1}{2n+1}x^{2n+1}.$$

From the ratio test, if $a_n = (-1)^n(n!)^{-1}\frac{1}{2n+1}x^{2n+1}$, then $|a_{n+1}/a_n| = \frac{1}{n+1}\frac{2n+3}{2n+1}|x|^2 \to 0$ as $n \to \infty$, so the series converges absolutely for all x, i.e. the radius of convergence is ∞. Then

$$F(1/\sqrt{2}) = \frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}(-1)^n(n!)^{-1}\frac{1}{2n+1}2^{-n-1/2}.$$

So, $F(1/\sqrt{2})$ is an alternating series of decreasing numbers. So, from the error bound for alternating series, if T_n denotes the nth Taylor polynomial at $x=0$, we have

$$\left|F(1/\sqrt{2}) - T_n(1/\sqrt{2})\right| \leq \frac{2}{\sqrt{\pi}}\frac{1}{(n+1)!}\frac{1}{2n+3}\frac{1}{2^{n+3/2}}.$$

So, summing $n=5$ terms should suffice, since then $\frac{2}{\sqrt{\pi}}\frac{1}{(n+1)!}\frac{1}{2n+3}\frac{1}{2^{n+3/2}} \approx 1.33\times 10^{-6}$. And indeed, we have $T_4(1/\sqrt{2}) \approx 0.682688$, so the first four decimal places of $F(1/\sqrt{2})$ are 0.6826 (without rounding).

Now

$$F(2/\sqrt{2}) = F(\sqrt{2}) = \frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}(-1)^n(n!)^{-1}\frac{1}{2n+1}2^{n+1/2}.$$

So, $F(2/\sqrt{2})$ is an alternating series of decreasing numbers. So, from the error bound for alternating series, if T_n denotes the nth Taylor polynomial at $x=0$, we have

$$\left|F(1/\sqrt{2}) - T_n(1/\sqrt{2})\right| \leq \frac{2}{\sqrt{\pi}}\frac{1}{(n+1)!}\frac{1}{2n+3}2^{n+3/2}.$$

So, summing $n = 6$ terms should suffice, since then $\frac{2}{\sqrt{\pi}} \frac{1}{(n+1)!} \frac{1}{2n+3} \frac{1}{2^{n+3/2}} \approx .0027$. And indeed, we have $T_6(1/\sqrt{2}) \approx 0.9567$, so the first two decimal places of $F(2/\sqrt{2})$ are 0.95 (without rounding).

Optional: Lastly,

$$F(3/\sqrt{2}) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} (-1)^n (n!)^{-1} \frac{1}{2n+1} (3/\sqrt{2})^{2n+1}.$$

Unfortunately, $F(3/\sqrt{2})$ is not an alternating series of decreasing numbers. So, to get an approximation for $F(3/\sqrt{2})$ without guaranteed error bounds, we simply sum up the first 20 terms of the series. We then get $T_{20}(3/\sqrt{2}) \approx .9973$.

In summary, $F(1/\sqrt{2})$ represents the area under the bell curve within one standard deviation of the mean (which is zero), $F(2/\sqrt{2})$ represents the area under the bell curve within two standard deviations of the mean, and $F(3/\sqrt{2})$ represents the area under the bell curve within three standard deviations of the mean. And we have the approximations $F(1/\sqrt{2}) \approx .68$, $F(2/\sqrt{2}) \approx .95$ and $F(3/\sqrt{2}) \approx .99$.

□

Exercise 10.4. Let $i = \sqrt{-1}$. Using the Maclaurin series for $\sin(x)$, $\cos(x)$ and e^x, verify Euler's identity

$$e^{ix} = \cos(x) + i \sin(x).$$

In particular, using $x = \pi$, we have

$$e^{i\pi} + 1 = 0.$$

Also, use Euler's identity to prove the following equalities

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2}.$$

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}.$$

In particular, we finally see that the hyperbolic sine and cosine functions are exactly the usual sine and cosine functions, evaluated on imaginary numbers.

$$\cos(x) = \cosh(ix).$$

$$\sin(x) = \sinh(ix)/i.$$

Solution. Recall that $e^x = \sum_{n=0}^{\infty} x^n/n!$, so

$$e^{ix} = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = \sum_{n \geq 0 \text{ even}} \frac{(-1)^{n/2} x^n}{n!} + \sum_{n \geq 0 \text{ odd}} \frac{i(-1)^{(n-1)/2} x^n}{n!} = \cos(x) + i \sin(x).$$

Setting $x = \pi$, we get

$$e^{i\pi} = \cos(\pi) + i \sin(\pi) = 1 + 0 \cdot i = 1.$$

Since $e^{ix} = \cos(x) + i \sin(x)$, we have $e^{-ix} = \cos(x) + i \sin(-x) = \cos(x) - i \sin(x)$. Adding and substracting these identities, we get

$$e^{ix} + e^{-ix} = 2 \cos x, \qquad e^{ix} - e^{-ix} = 2i \sin x.$$

So, recalling that $2\cosh(x) = e^x + e^{-x}$ and $2\sinh(x) = e^x - e^{-x}$, we have

$$2\cosh(ix) = e^{ix} + e^{-ix} = 2\cos(x), \qquad 2\sinh(ix)/i = (e^{ix} - e^{-ix})/i = 2\sin(x).$$

□

Exercise 10.5. Euler's identity can be used to remember all of the multiple angle formulas that are easy to forget. For example, note that

$$\cos(2x) + i\sin(2x) = e^{2ix} = (e^{ix})^2 = (\cos(x) + i\sin(x))^2 = \cos^2(x) - \sin^2(x) + 2i\sin(x)\cos(x).$$

By equating the real and imaginary parts of this identity, we therefore get

$$\cos(2x) = \cos^2(x) - \sin^2(x)$$

$$\sin(2x) = 2\sin(x)\cos(x).$$

Derive the triple angle identities in this same way, using $e^{3ix} = (e^{ix})^3$.

Solution. Note that $\cos(3x) + i\sin(3x) = e^{3ix} = (e^{ix})^3 = (\cos(x) + i\sin(x))^3 = \cos^3(x) + 3i\cos^2(x)\sin(x) - 3\cos(x)\sin^2(x) - i\sin^3(x)$. Equating the real and imaginary parts, we get $\cos(3x) = \cos^3(x) - 3\cos(x)\sin^2(x)$, and $\sin(3x) = 3\cos^2(x)\sin(x) - \sin^3(x)$. □

Exercise 10.6. Find the sum of the infinite series

$$1 + \frac{2}{10} + \frac{3}{10^2} + \frac{7}{10^3} + \frac{2}{10^4} + \frac{3}{10^5} + \frac{7}{10^6} + \frac{2}{10^7} + \frac{3}{10^8} + \frac{7}{10^9} + \cdots.$$

Solution. In decimal notation, the series is $1.237237237237237\ldots$. In fraction notation, the series is $1 + 237/999$, which is $1236/999$. □

Exercise 10.7. Using Maclaurin series, evaluate the following limit

$$\lim_{x\to 0} \frac{\cos(x) - 1 + x^2/2}{x^4}.$$

Solution. Recall that $\cos(x) = 1 - x^2/2 + x^4/4! - x^6/6! - \cdots$, so $(\cos(x) - 1 + x^2/2)/x^4 = (1/4!) + x^2/6! - \cdots$, so letting $x \to 0$, we get $\lim_{x\to 0}(\cos(x) - 1 + x^2/2)/x^4 = 1/24$. □

11. Homework 11

Exercise 11.1. Consider the parametric curve $s\colon \mathbf{R} \to \mathbf{R}^2$ defined for any real t by

$$s(t) = (t^2 + 1, t^3 + t).$$

Find the tangent line to the curve when $t = 2$. You can express this line either in parametric form, or in the form $y = mx + b$. What is the slope of the tangent line?

Write a formula for the arc length of the curve between $t = 0$ and $t = 3$. You do not have to evaluate this integral.

Solution. We have $s'(t) = (2t, 3t^2 + 1)$, so $s'(2) = (4, 13)$, and the tangent line is of the form

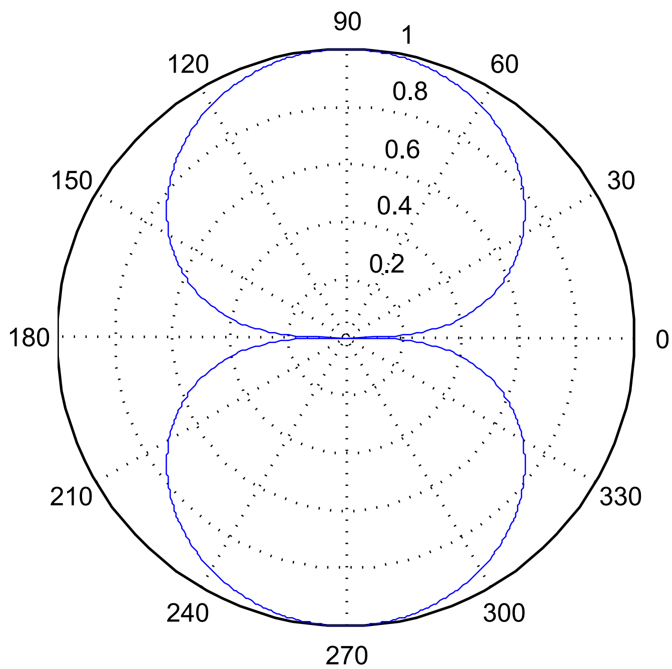
$$h(t) = s(2) + ts'(2) = (5, 10) + t(4, 13).$$

The arc length is

$$\int_0^3 ||s'(t)||\,dt = \int_0^3 \sqrt{(2t)^2 + (3t^2+1)^2}dt.$$

□

Exercise 11.2. Using polar coordinates (r, θ), plot the function $r^2 = \sin\theta$ for $\theta \in (0, \pi)$. The end result should resemble a figure eight. Compute the area enclosed by the curve. Write a formula that computes the length of the curve [you do NOT have to evaluate this integral].



Solution. The area is

$$2\int_0^\pi (1/2)(r(\theta))^2 d\theta = \int_0^\pi \sin\theta d\theta = 2.$$

The arc length is

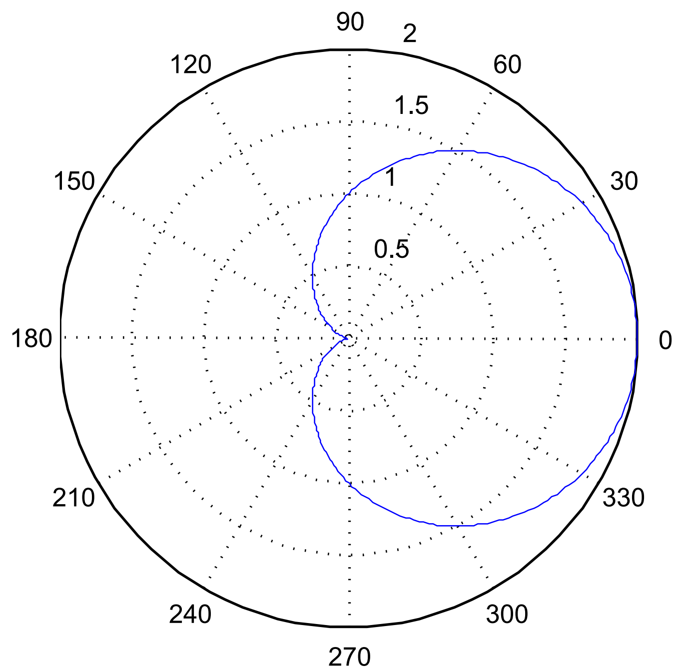
$$2\int_0^\pi \sqrt{(r(\theta))^2 + (r'(\theta))^2}d\theta 2\int_0^\pi \sqrt{\sin\theta + (\cos\theta/(2\sqrt{\sin\theta}))^2}d\theta.$$

□

Exercise 11.3. Using polar coordinates (r, θ), plot the function $r = 1+\cos\theta$. The end result should resemble an apple, or a heart. Compute the area enclosed by the curve. Compute the length of the curve.

Solution. The area is

$$\int_0^{2\pi} (1/2)(r(\theta))^2 d\theta = (1/2)\int_0^{2\pi} 1 + \cos^2\theta + 2\cos\theta d\theta = (1/2)\int_0^{2\pi} 1 + (1/2)d\theta = 3\pi/2.$$



The arc length is

$$\begin{aligned}\int_0^{2\pi}\sqrt{(r(\theta))^2+(r'(\theta))^2}d\theta &= \int_0^{2\pi}\sqrt{1+\cos^2\theta+2\cos\theta+(\sin\theta)^2}d\theta \\ &= \int_0^{2\pi}\sqrt{2}\sqrt{2\cos^2(\theta/2)}d\theta = 2\int_0^{2\pi}\left|\cos(\theta/2)\right|d\theta \\ &= 4\int_0^{\pi}\left|\cos(\theta/2)\right|d\theta = 4\int_0^{\pi}\cos(\theta/2)d\theta = 8\sin(\theta/2)|_0^{\pi} = 8.\end{aligned}$$

□

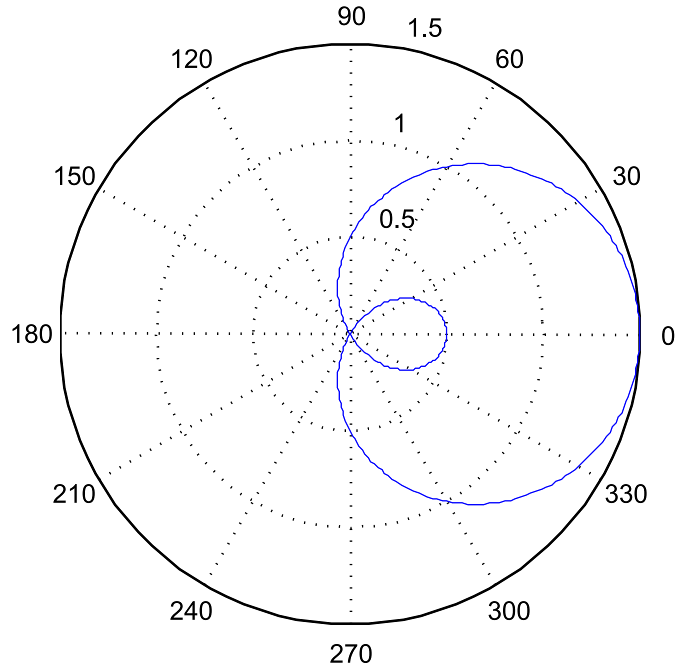
Exercise 11.4. Using polar coordinates (r,θ), plot the function $r=(1/2)+\cos\theta$. Find the tangent line to the curve when $\theta=\pi/4$. You can express this line either in parametric form, or in the form $y=mx+b$. What is the slope of the tangent line?

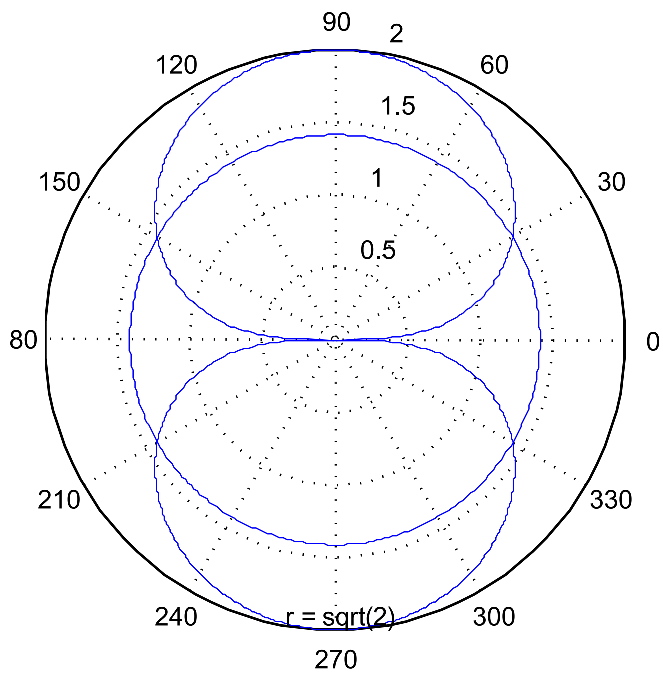
Solution. We have $s(\theta)=(r(\theta)\cos\theta, r(\theta)\sin(\theta))$ as the parametrization of the curve in Euclidean coordinates, and $s'(\theta)=(-r\sin\theta+r'\cos\theta, r\cos\theta+r'\sin\theta)$, so $s'(\pi/4)=(-[(1+\sqrt{2})/2](\sqrt{2}/2)-(\sqrt{2}/2)^2,\ [(1+\sqrt{2})/2](\sqrt{2}/2)-(\sqrt{2}/2)^2)$ the tangent line is

$$\begin{aligned}h(\theta) &= s(\pi/4)+\theta s'(\pi/4) \\ &= ([(1+\sqrt{2})/2](\sqrt{2}/2),\ [(1+\sqrt{2})/2](\sqrt{2}/2),\ ) \\ &\quad +\theta(-[(1+\sqrt{2})/2](\sqrt{2}/2)-(\sqrt{2}/2)^2,\ [(1+\sqrt{2})/2](\sqrt{2}/2)-(\sqrt{2}/2)^2).\end{aligned}$$

The slope of the tangent line is $\frac{[(1+\sqrt{2})/2](\sqrt{2}/2)-(\sqrt{2}/2)^2}{-[(1+\sqrt{2})/2](\sqrt{2}/2)-(\sqrt{2}/2)^2}$. □

Exercise 11.5. Using polar coordinates (r,θ), plot the functions $r=\sqrt{2}$ and $r^2=4\sin\theta$, and label their points of intersection. Find the tangent line to each curve when $\theta=\pi/4$. You can express the lines either in parametric form, or in the form $y=mx+b$. What is the slope of each tangent line?





Solution. Solving for the points of intersection, we get $2 = 4\sin\theta$, so $\sin\theta = 1/2$, so $\theta = \pi/6$ and $5\pi/6$. Since both curves are preserved by changing r to $-r$, we also have points of intersection at $\theta = 7\pi/6$ and $11\pi/6$. (At all points of intersection, $r = \sqrt{2}$.)

We have $s(\theta) = (r(\theta)\cos\theta, r(\theta)\sin(\theta))$ as the parametrization of the curves in Euclidean coordinates, and $s'(\theta) = (-r\sin\theta + r'\cos\theta, r\cos\theta + r'\sin\theta)$, so for $r = 2\sqrt{\sin\theta}$, $s'(\pi/4) = (-2^{3/4}2^{-1/2} + 2^{-1/4}2^{-1/2},\ 2^{3/4}2^{-1/2} + 2^{-1/4}2^{-1/2})$ the tangent line is

$h(\theta) = s(\pi/4) + \theta s'(\pi/4) = (2^{3/4}2^{-1/2},)2^{3/4}2^{-1/2},+\theta(-2^{3/4}2^{-1/2}+2^{-1/4}2^{-1/2},\ 2^{3/4}2^{-1/2}+2^{-1/4}2^{-1/2})$.

The slope of the tangent line is $\frac{2^{3/4}2^{-1/2}+2^{-1/4}2^{-1/2}}{-2^{3/4}2^{-1/2}+2^{-1/4}2^{-1/2}}$. □

Exercise 11.6. Suppose you have a heavy chain that is 50 kg and 2 meters long. The chain is resting on the ground. Calculate the work required to lift one end of the chain:

- 1 meter off the ground.
- 2 meters off the ground.
- 3 meters off the ground.

Ignore any effects of friction between the chain and the ground.

Solution.

- 1 meter off the ground: $g\int_0^1(1-x)25dx$
- 2 meters off the ground. $g\int_0^2(2-x)25dx$
- 3 meters off the ground. $g\int_0^2(3-x)25dx$

□

Exercise 11.7. Suppose there is an underground oil deposit. The oil has density ρ. The oil deposit is contained in a cavern that is the shape of a sphere of radius 100 meters. The center of the sphere is 500 meters below the surface of the earth. The sphere is completely full of oil. Determine the work require to pump all of the oil to the earth's surface.

Solution. At depth $400 < z < 600$, a thin horizontal slice of the sphere is a disk described by $x^2 + y^2 = 100^2 - (z - 500)^2$, so it has a radius $\sqrt{100^2 - (z - 500)^2}$. This thin slice of liquid is transported a distance of z. So, the work required is

$$\rho g \int_{400}^{600} z\sqrt{100^2 - (z - 500)^2}dz = \rho g 2500000\pi.$$

□

The Exercises below are OPTIONAL. The Exercises below will NOT be covered on any quiz. These exercises are meant to help with your final exam studying.

Exercise 11.8 (Optional)**.** If the following limit exists, calculate it:

$$\lim_{x\to 0}(1+\sin(3x^2))^{\frac{1}{6x^2}}.$$

Solution. Let L be the limit. Then by continuity of ln,

$$\ln L = \lim_{x\to 0}\ln(1+\sin(3x^2))^{\frac{1}{6x^2}} = \lim_{x\to 0}\frac{\ln(1+\sin(3x^2))}{6x^2} \overset{L'H}{=} \lim_{x\to 0}\frac{6x\cos(3x^2)}{[1+\sin(3x^2)]12x} = \lim_{x\to 0}\frac{\cos(3x^2)}{2[1+\sin(3x^2)]} = \frac{1}{2}.$$

In the last equality, the function is continuous in x, so we can just "plug in" to get the answer. □

Exercise 11.9 (Optional)**.**

- Evaluate $\int_0^{\pi/12} \sin^2(3x)dx$.
- Calculate $\int \frac{\sqrt{1-x^2}}{x^2}dx$. (Hint: $\cot'(x) = -\csc^2 x$)

Solution.

$$\int_0^{\pi/12} \sin^2(3x)dx = \int_0^{\pi/12} (1/2)(1-\cos(6x))dx = (1/2)(x-(1/6)\sin(6x))|_0^{\pi/12} = (1/2)(\pi/12-1/6).$$

Using the substitution $x = \sin\theta$, $dx = \cos\theta$, (noting that $-\pi/2 \le \theta \le \pi/2$, so $\cos\theta \ge 0$)

$$\begin{aligned}\int \frac{\sqrt{1-x^2}}{x^2}dx &= \int \frac{\sqrt{\cos^2\theta}}{\sin^2\theta}\cos\theta\, d\theta = \int \frac{\cos^2}{\sin^2\theta}\, d\theta = \int \cos^2\theta \csc^2\theta\, d\theta \\ &= \int \cos^2\theta(-\cot'\theta)\, d\theta = -\cos^2\theta\cot\theta - \int 2\cos\theta\sin\theta\cot\theta\, d\theta \\ &= -\cos^3\theta/\sin\theta - \int 2\cos^2\theta\, d\theta = -\cos^3\theta/\sin\theta - \int (1+\cos(2\theta))\, d\theta \\ &= -\cos^3\theta/\sin\theta - \theta - (1/2)\sin(2\theta) + C.\end{aligned}$$

Using $\theta = \sin^{-1}(x)$, $\cos(\sin^{-1}(x)) = \sqrt{1-x^2}$ and $\sin(2\theta) = 2\sin\theta\cos\theta$, we get

$$\int \frac{\sqrt{1-x^2}}{x^2}dx = -(1-x^2)^{3/2}/x - \sin^{-1}(x) - x\sqrt{1-x^2} + C. = -\sqrt{1-x^2}/x - \sin^{-1}(x) + C.$$

□

Exercise 11.10 (Optional)**.** Write the following function as a partial fraction.

$$\frac{3x^2-5x+8}{(x+1)(x^2-2x+5)}$$

Solution. The factor x^2-2x+5 does not have real roots since $2^2-4(1)(5) < 0$, so we write

$$\frac{3x^2-5x+8}{(x+1)(x^2-2x+5)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-2x+5},$$

and solve for A, B, C. That is, we solve

$$3x^2-5x+8 = A(x^2-2x+5)+(Bx+C)(x+1) = x^2(A+B)+x(-2A+C+B)+(5A+C).$$

So, $A+B=3$, $-2A+C+B=-5$ and $5A+C=8$. Adding copies of the first equation to the second and third, we get

$$A+B=3, \quad C+3B=1, \quad -5B+C=-7.$$

Subtracting the second equation from the third,

$$A+B=3, \quad C+3B=1, \quad -8B=-8.$$

So, $B=1$, $C=1-3B=1-3=-2$, and $A=3-B=3-1=2$. That is

$$\frac{3x^2-5x+8}{(x+1)(x^2-2x+5)} = \frac{2}{x+1} + \frac{x-2}{x^2-2x+5}.$$

□

Exercise 11.11 (Optional)**.** Decide whether or not the following integrals converge or diverge. You do not have to compute the values of the integrals.

- $$\int_0^\infty \frac{2\cos(x^4)}{x^5+\sqrt{x}+1}$$
- $$\int_0^3 \frac{10}{x(x^2+2)}$$

Solution. For any $x \geq 1$, we have

$$\left|\frac{2\cos(x^4)}{x^5+\sqrt{x}+1}\right| \leq \frac{2}{x^5} = 2x^{-5}$$

And for any $0 < x < 1$,

$$\left|\frac{2\cos(x^4)}{x^5+\sqrt{x}+1}\right| \leq \frac{2}{\sqrt{x}} = 2x^{-1/2}.$$

So,

$$\left|\int_0^\infty \frac{2\cos(x^4)}{x^5+\sqrt{x}+1}dx\right| \leq \int_0^\infty \left|\frac{2\cos(x^4)}{x^5+\sqrt{x}+1}\right| dx \leq \int_0^1 2x^{-1/2}dx + \int_1^\infty 2x^{-5}dx < \infty.$$

Therefore, the first integral converges.

The second integral diverges, since for any $0 < x < 3$, $x^2+2 \leq 11$, so

$$\frac{10}{x(x^2+2)} \geq \frac{10}{x(11)},$$

so

$$\int_0^3 \frac{10}{x(x^2+2)} \geq \frac{10}{11}\int_0^3 \frac{1}{x}dx = \infty.$$

□

Exercise 11.12 (Optional)**.** Compute the volume of the solid of revolution obtained by rotating the region D bounded by the curves $x = 0$, $y = \ln(x)$, and $y = 0$ about the line $x = -1$.

Solution. The volume is given by e.g. the method of cylindrical shells:

$$\begin{aligned} 2\pi \int_0^1 (x-(-1))\,|\ln(x)|\,dx &= 2\pi \int_0^1 (x+1)\ln(1/x)dx = 2\pi \int_0^1 [x^2/2+x]' \ln(1/x)dx \\ &= 2\pi[\ln(1/x)(x^2/2+x)]_0^1 - \int_0^1 [x^2/2+x][\ln(1/x)]' dx \\ &= 2\pi \lim_{s\to 0^+}[\ln(1/x)(x^2/2+x)]_s^1 - \int_0^1 [x^2/2+x]x\cdot(-x^{-2})dx \\ &= 2\pi \lim_{s\to 0^+}[-\ln(1/s)(s^2/2+s)] - \int_0^1 [x^2/2+x]x\cdot(-x^{-2})dx \\ &= 2\pi \int_0^1 [x/2+1]dx = 2\pi[x+x^2/4]_0^1 = 2\pi[1+1/4] = 5\pi/2. \end{aligned}$$

We used here that $\lim_{s\to 0^+} s\ln(1/s) = 0$ and $\lim_{s\to 0^+} s^2\ln(1/s) = 0$, which follows by L'Hôpital's rule. □

Exercise 11.13 (Optional)**.** Determine whether or not the following series converge or diverge.

- $$\sum_{n=4}^{\infty}(-1)^n \frac{n}{n^2+4}$$
- $$\sum_{n=6}^{\infty}(-1)^n \Big(\frac{4e^{2n}-4}{5e^n+6e^{2n}}\Big)^{n-1}$$
- $$\sum_{n=2}^{\infty} 2n\sin(2/n).$$
- $$\sum_{n=5}^{\infty} \ln\Big(1+\frac{1}{n^3}\Big).$$

Solution. The first sum converges because it is an alternating series of decreasing positive numbers. That is, it is of the form $\sum_{n=4}^{\infty}(-1)^n a_n$, where $a_4 > a_5 > a_6 > \cdots$ and $\lim_{n\to\infty} a_n = 0$, where $a_n = n/(n^2+4)$. Indeed,

$$\frac{a_{n+1}}{a_n} = \frac{n+1}{n}\frac{n^2+4}{(n+1)^2+4} = \frac{n+1}{n}\frac{n^2+4}{n^2+4+2n} = \frac{n^3+n^2+4n+1}{n^3+2n^2+4n} < 1,$$

since $n^2 > 1$ when $n \geq 4$.

For the second series, we use the ratio test to get

$$\lim_{n\to\infty}\left|(-1)^n\Big(\frac{4e^{2n}-4}{5e^n+6e^{2n}}\Big)^{n-1}\right|^{1/n} = \lim_{n\to\infty}\left|\frac{4e^{2n}-4}{5e^n+6e^{2n}}\right|^{1-1/n} = \lim_{n\to\infty}\frac{4}{6} = \frac{2}{3} < 1.$$

Since this limit is less than 1, the ratio test says the sum converges.

The third series diverges by the divergence test, since $\lim_{n\to\infty} 2n\sin(2/n) = 4 \neq 0$. (This follows either by Taylor expanding the sin function, or by L'Hôpital's rule.)

For the fourth sum, note that, by definition of ln,

$$0 \leq \ln(1+n^{-3}) = \int_1^{1+n^{-3}} x^{-1}dx \leq n^{-3} \max_{1\leq x\leq 1+n^{-3}} x^{-1} = n^{-3}.$$

So, by comparison,

$$0 \leq \sum_{n=5}^{\infty} \ln\left(1+\frac{1}{n^3}\right) \leq \sum_{n=5}^{\infty} n^{-3} < \infty.$$

The last sum converges by e.g. the integral test, since $\int_4^\infty x^{-3}dx < \infty$. So, the fourth sum converges. □

Exercise 11.14 (Optional)**.** Define $f(x) = \frac{4}{5-x}$ for any real x.

- Show that f is an increasing function, and for any x in $[0,3]$, $f(x)$ is also in $[0,3]$.
- Let $b_0 = 5/2$, and for any $n \geq 1$ define $b_n = f(b_{n-1})$. Show that, for any $n \geq 0$, $0 \leq b_n \leq 3$. Then show that the sequence $b_0, b_1, \ldots$ is decreasing.
- Show that the sequence $b_0, b_1, \ldots$ converges to a limit L. Find the value of L.

Solution. Note that $f'(x) = 4/(5-x)^2 > 0$ so f is always increasing. Also, $f(0) = 4/5$ and $f(3) = 4/2 = 2$, so $0 < x < 3$ implies that $4/5 < f(x) < 2$, so that $0 < f(x) < 3$ holds for any x in $[0,3]$.

We now show that $0 \leq b_n \leq 3$ by induction. The base case holds since $b_0 \in [0,3]$. We now induct on n. We assume $b_n \in [0,3]$ and we need to show that $b_{n+1} \in [0,3]$. Recall that f is an increasing function on $[0,3]$. That is, for any $x \in [0,3]$, $f(0) \leq f(x) \leq f(3)$. That is, for any $x \in [0,3]$, $4/5 \leq f(x) \leq 2$. By the inductive hypothesis, $b_n \in [0,3]$, so that $b_{n+1} = f(b_n) \in [4/5,2] \subseteq [0,3]$. Having completed the inductive step, the assertion follows by induction.

We now show that $b_0, b_1, \ldots$ is decreasing by induction. The base case holds since $b_0 = 5/2$ and $b_1 = 4/(5-5/2) = 4/(5/2) = 8/5 < b_0$. We now induct on n. We assume $b_{n+1} < b_n$ and we need to show that $b_{n+2} < b_{n+1}$. Recall that f is an increasing function on $[0,3]$. Since $b_{n+1} < b_n$, we therefore have $f(b_{n+1}) < f(b_n)$. By definition of b_{n+1}, this says that $b_{n+2} < b_{n+1}$. That is, we have completed the inductive step. Having completed the inductive step, the assertion follows by induction.

The sequence $b_0, b_1, \ldots$ is monotonic and bounded by part(b). It therefore converges. Let $L = \lim_{n\to\infty} b_n$. Then, using the definition of b_{n+1} and the continuity of f on $[0,3]$,

$$L = \lim_{n\to\infty} b_n = \lim_{n\to\infty} b_{n+1} = \lim_{n\to\infty} f(b_n) = f(\lim_{n\to\infty} b_n) = f(L).$$

That is, $L = f(L)$. That is, $L = 4/(5-L)$, i.e. $-L^2 + 5L - 4 = 0$, i.e. $L = [-5 \pm \sqrt{25-16}]/(-2) = (5/2) \pm 3/2$. That is, $L = 4$ or $L = 1$. Since $b_0 = 5/2$ and the sequence is decreasing, we conclude that $L = 1$. □

Exercise 11.15 (Optional)**.** For any real x, define

$$f(x) = \sum_{n=1}^{\infty} (-1)^{n+1} n x^{3n+1}.$$

- Find the interval of convergence of f. That is, find all points where the sum converges.
- Find a simple formula for f (that does not involve any $\sum$ signs).

Solution. Let $a_n = (-1)^{n+1} n x^{3n+1}$. From the ratio test,

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)x^{3n+4}}{nx^{3n+1}}\right| = \lim_{n\to\infty} \left|\frac{n+1}{n}\right| |x|^3 = |x|^3.$$

So, the ratio test implies that the sum converges when $|x| < 1$ and it diverges when $|x| > 1$. To check the convergence when $x = \pm 1$ we just plug in. We get

$$f(1) = \sum_{n=1}^{\infty} (-1)^{n+1} n, \qquad f(-1) = \sum_{n=1}^{\infty} (-1)^{4n+2} n.$$

Both sums diverge, by the divergence test. So, the interval of convergence is $(-1, 1)$, and the sum diverges elsewhere.

To find a formula for f, recall that, for any $|x| < 1$,

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

So

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n$$

$$\frac{1}{1+x^3} = \sum_{n=0}^{\infty} (-1)^n x^{3n}$$

Taking derivatives of both sides (which can be done term-by-term within the radius of convergence)

$$\frac{-3x^2}{(1+x^3)^2} = \sum_{n=0}^{\infty} (-1)^n 3n x^{3n-1}$$

Multiplying both sides by $-x^2/3$ then gives

$$\frac{x^4}{(1+x^3)^2} = \sum_{n=0}^{\infty} (-1)^{n+1} n x^{3n+1}$$

□

Exercise 11.16 (Optional)**.** Approximate the value of the integral

$$\int_0^1 \frac{x}{3+x^3} dx$$

within four decimal places of accuracy.

Solution. When $|x| < 1$ we have by Taylor series (and the previous problem)

$$\frac{1}{1+x^3} = \sum_{n=0}^{\infty}(-1)^n x^{3n}$$

$$\frac{1}{3+3x^3} = \frac{1}{3}\sum_{n=0}^{\infty}(-1)^n x^{3n}$$

$$\frac{1}{3+x^3} = \frac{1}{3}\sum_{n=0}^{\infty}(-1)^n (x/3^{1/3})^{3n} = \frac{1}{3}\sum_{n=0}^{\infty}(-1)^n 3^{-n} x^{3n}$$

$$\frac{x}{3+x^3} = \frac{1}{3}\sum_{n=0}^{\infty}(-1)^n 3^{-n} x^{3n+1}.$$

The first series converges whenever $|x| < 1$. Since we replaced x with $x/3^{1/3}$, the last series converges whenever $|x| < 3^{1/3}$. Integrating both sides term by term (which is allowed when $0 \leq x \leq 1$, since this is inside the radius of convergence), we get

$$\int_0^1 \frac{x}{3+x^3}dx = \frac{1}{3}\sum_{n=0}^{\infty}(-1)^n 3^{-n}\int_0^1 x^{3n+1}dx = \frac{1}{3}\sum_{n=0}^{\infty}(-1)^n 3^{-n}\frac{1}{3n+2}.$$

This series S is an alternating series. Consider its partial sum

$$S_N = \sum_{n=0}^{N}(-1)^n 3^{-n-1}\frac{1}{3n+2}.$$

The error bound for alternating series says that

$$|S - S_N| < a_{N+1},$$

where $a_N = 3^{-N-1}\frac{1}{3N+2}$. So, we just have to find N such that $a_{N+1} < 10^{-5}$. This holds when $N = 6$, since $3^{-8}/23 < 10^{-5}$. And

$$S_6 = \frac{1}{3(2)} - \frac{1}{9(5)} + \frac{1}{27(8)} - \frac{1}{81(11)} + \frac{1}{243(14)} - \frac{1}{729(17)} \approx .1481$$

In conclusion, the first four decimal places of the integral $\int_0^1 \frac{x}{3+x^3}dx$ are: .1481

□

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