Please provide complete and well-written solutions to the following exercises.

Assignment 9

Due December 4, at the beginning of class.

Exercise 1. Let $f(x,y,z) = x^2 + 2yz$. Suppose $x = ab$, $y = bc$ and $z = a^2bc^2$. Compute $\partial f/\partial a$ directly, and also compute this derivative using the Chain Rule.

Exercise 2. Let $f\colon \mathbf{R}^2 \to \mathbf{R}$. Polar coordinates r,θ are defined by $x = r\cos\theta$ and $y = r\sin\theta$. Rewrite f_x just in terms of derivatives of f with respect to r and with respect to θ. Also, the expression for f_x should only be in terms of r and θ.

Exercise 3 (Additional challenge, ungraded)**.** Recall that the equation $\frac{\partial^2}{\partial x^2} f + \frac{\partial^2}{\partial y^2} f = 0$ is important in many applications. Sometimes it is convenient to write this equation in polar coordinates instead. Using the Chain Rule, convert the equation $\frac{\partial^2}{\partial x^2} f + \frac{\partial^2}{\partial y^2} f = 0$ into polar coordinates. That is, rewrite this expression as an equality just in terms of derivatives of f with respect to r and with respect to θ. (Hint: it is probably easier to write the expression $\frac{\partial^2 f}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{1}{r}\frac{\partial f}{\partial r}$ in terms of derivatives of x and y; in the end this expression should be equal to $\frac{\partial^2}{\partial x^2} f + \frac{\partial^2}{\partial y^2} f$)

Exercise 4. Suppose $x^2 + 2yz + z^3 - 5 = 0$. Compute $\partial y/\partial z$.

Exercise 5. Let $f(x,y) = x^2 + xy + y$. Identify the critical points of f, and identify these points as local maxima, local minima or saddle points.

Exercise 6. Find the maximum and minimum values of the function $f(x,y) = x^2 + yx$ on the disk $x^2 + y^2 \leq 1$.

Exercise 7. Find the maximum and minimum values of the function $f(x,y) = e^{-x^2-y^2}$ in the plane, if they exist. If they do not exist, briefly explain why.

Exercise 8. Suppose a have a circular cylinder of the form $x^2 + y^2 \leq r^2$, where $0 \leq z \leq h$. Suppose this cylinder has constant volume 1. Find the maximum and minimum values of the surface area of the cylinder, if they exist. If they do not exist, briefly explain why.

Exercise 9. Unfortunately, the second derivative test as we stated in this class does not really work anymore for more than two variables. There is a way to modify the test to work for more than two variables, but it requires an understanding of linear algebra, so we cannot state this modification in this class. Let's see what goes wrong. Consider the function $f(x,y,z) = -x^2 - y^2 + z^2$. Show that the point $(0,0,0)$ is neither a local maximum nor a local minimum of f. Nevertheless, show that $\nabla f(0,0,0) = (0,0,0)$, and

$$\det\begin{pmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{pmatrix} > 0.$$

Exercise 10. In statistics and other applications, we can be presented with data points $(x_1, y_1), \ldots, (x_n, y_n)$. We would like to find the line $y = mx + b$ which lies "closest" to all of these data points. Such a line is known as a **linear regression**. There are many ways to define the "closest" such line. The standard method is to use **least squares minimization**. A line which lies close to all of the data points should make the quantities $(y_i - mx_i - b)$ all very small. We would like to find numbers m, b such that the following quantity is minimized:

$$f(m, b) = \sum_{i=1}^{n} (y_i - mx_i - b)^2.$$

Using the second derivative test, show that the minimum value of f is achieved when

$$m = \frac{\left(\sum_{i=1}^{n} x_i\right)\left(\sum_{j=1}^{n} y_j\right) - n\left(\sum_{k=1}^{n} x_k y_k\right)}{\left(\sum_{i=1}^{n} x_i\right)^2 - n\left(\sum_{j=1}^{n} x_j^2\right)}.$$

$$b = \frac{1}{n}\left(\sum_{i=1}^{n} y_i - m \sum_{j=1}^{n} x_j\right).$$

Briefly explain why this is actually the minimum value of $f(m, b)$. (You are allowed to use the inequality $(\sum_{i=1}^{n} x_i)^2 \leq n(\sum_{i=1}^{n} x_i^2)$.)

Exercise 11. Let $f(x, y) = x^2 + y^2$. Using Lagrange Multipliers, find the maximum and minimum of f on the ellipse $x^2 + 2y^2 = 1$.

Exercise 12. Let $f(x, y, z) = x^2 + zy$. Using Lagrange Multipliers, find the maximum and minimum of f on the sphere $x^2 + y^2 + z^2 = 1$.

Exercise 13. Let a, b, c be positive constants. An ice cream cone is defined as the surface $z = a\sqrt{x^2 + y^2}$ where $z \leq b$. Suppose the ice cream cone has surface area c. Using Lagrange Multipliers, find the ice cream cone of fixed surface area c and with maximum volume. (This way, you get to eat the most ice cream with the least amount of material.) You should probably use the variables r and b, where r denotes the radius of the cone, and where b is the height of the cone. You can freely use that the surface area of the cone is $\pi r\sqrt{r^2 + b^2}$. (Deriving this surface area formula requires things from Math 32B.)