2: DERIVATIVES

STEVEN HEILMAN

1. DEFINITION OF THE DERIVATIVE

And I, infinitesimal being, drunk with the great starry void, likeness, image of mystery, I felt myself a pure part of the abyss, I wheeled with the stars, my heart broke loose on the wind.
  Pablo Neruda, 100 Love Sonnets.

The derivative is one of the central concepts in calculus. It is also one of the most useful. For a function $f\colon \mathbb{R} \to \mathbb{R}$, where $f(t)$ denotes the vertical position of a falling object at time t, the derivative of t is the velocity of the object. For a general function $f\colon \mathbb{R} \to \mathbb{R}$, the derivative of f is the rate of change of f. We denote the derivative of f at x by $f'(x)$. One particular use for the derivative is in optimization. If $f\colon \mathbb{R} \to \mathbb{R}$ is a cost function, then the maximum and minimum values of f occur when $f'(x) = 0$. So, if we want to maximize or minimize f, we only need to find the zeros of $f'(x)$.

We now begin the construction of the derivative. Let $f\colon \mathbb{R} \to \mathbb{R}$. Suppose $(x_0, y_0), (x_1, y_1) \in \mathbb{R} \times \mathbb{R}$, $y_0 = f(x_0)$ and $y_1 = f(x_1)$. Recall that the line through the points $(x_0, y_0), (x_1, y_1)$ has slope $\frac{y_1 - y_0}{x_1 - x_0}$. So, if we let x_1 approach x_0, and if the number $\frac{y_1 - y_0}{x_1 - x_0}$ approaches some value L, then we can think of L as the infinitesimal rate of change of f. To see this approximation procedure in action, see the following MIT JAVA applet: secant approximation.

Definition 1.1. (The Derivative) Let $a, b, x \in \mathbb{R}$, and let $f\colon (a, b) \to \mathbb{R}$. We say that f is **differentiable** at x if the following limit exists.

$$\lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = f'(x) = \frac{df}{dx} = \frac{df}{dx}(x) = \frac{d}{dx} f(x)$$

If $f'(x)$ exists, we call $f'(x)$ the **derivative** of f at x. We say that f is **differentiable** on (a, b) if f is differentiable at every $x \in (a, b)$.

To observe some derivatives being calculated, see the following MIT JAVA applet: creating derivatives.

Since we have introduced a new concept, we now want to answer some basic questions to try to understand this new concept. We also try to relate this new concept to other concepts that we have already learned. We now show that our new notion of differentiability implies our old notion of continuity.

Proposition 1.2. Let $f\colon \mathbb{R} \to \mathbb{R}$ be differentiable on $\mathbb{R}$. Then f is continuous on $\mathbb{R}$.

Proof. Let $a \in \mathbb{R}$. Since f is differentiable at a, $\lim_{x \to a} \frac{f(x)-f(a)}{x-a}$ exists. For $x \neq a$, write $f(x) - f(a) = \frac{f(x)-f(a)}{x-a}(x - a)$. Then our limit law for products applies, yielding

$$\lim_{x \to a}(f(x) - f(a)) = \lim_{x \to a}\left(\frac{f(x) - f(a)}{x - a}(x - a)\right)$$

$$= \left(\lim_{x \to a}\frac{f(x) - f(a)}{x - a}\right)\left(\lim_{x \to a}(x - a)\right) . = f'(x) \cdot 0 = 0$$

So, $\lim_{x \to a} f(x) = f(a)$, i.e. f is continuous at a. $\square$

2. Subtleties of the Derivative

Is there a continuous function that is not differentiable at one point? In other words, does the converse to Proposition 1.2 hold?

Example 2.1. (A Discontinuous Derivative) Consider the function $f(x) = |x|$, $x \in \mathbb{R}$. Since $f(x) = x$ for $x > 0$, and $f(x) = -x$ for $x < 0$, we see that $f'(x)$ resembles the Heaviside function

$$f'(x) = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$$

But what happens at zero? Observe,

$$\lim_{h \to 0^+} \frac{f(0 + h) - f(0)}{h} = \lim_{h \to 0^+} \frac{h}{h} = \lim_{h \to 0^+} 1 = 1,$$

$$\lim_{h \to 0^-} \frac{f(0 + h) - f(0)}{h} = \lim_{h \to 0^-} \frac{-h}{h} = \lim_{h \to 0^-} (-1) = -1.$$

So, $f'(0)$ does not exist, and the converse of Proposition 1.2 is false. In fact, a stronger statement holds (see Problem 5.5).

From the contrapositive of Proposition 1.2, we know that if f is discontinuous, then f is not differentiable. There are even more ways for f to not be differentiable.

Example 2.2. (A Derivative Approaching Infinity) Let $x \in \mathbb{R}$, and let $f(x) = x^{1/3}$. For $x \neq 0$, $f'(x) = (1/3)x^{-2/3}$. So, $\lim_{x \to 0} f'(x) = \infty$, i.e. $\lim_{x \to 0} f'(x)$ DNE, i.e. $f(x)$ is not differentiable at $x = 0$.

3. Properties of the Derivative

(a) For $c \in \mathbb{R}$, and $f(x) = c$, we have $f'(x) = 0$.
(b) For $n \in \mathbb{R}$, and $f(x) = x^n$, we have $f'(x) = nx^{n-1}$ (for x such that x^n and nx^{n-1} are defined).
(c) For $c \in \mathbb{R}$ and $f \colon \mathbb{R} \to \mathbb{R}$ differentiable, $\frac{d}{dx}[cf(x)] = c\frac{d}{dx}f(x)$.
(d) For $f, g \colon \mathbb{R} \to \mathbb{R}$ differentiable, $\frac{d}{dx}[f(x)+g(x)] = [\frac{d}{dx}f(x)]+[\frac{d}{dx}g(x)]$, $\frac{d}{dx}[f(x)-g(x)] = [\frac{d}{dx}f(x)] - [\frac{d}{dx}g(x)]$.
(e) (**Product rule**) If $f, g \colon \mathbb{R} \to \mathbb{R}$ are differentiable, then

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + g'(x)f(x).$$

(f) (**Quotient rule**) If $f, g \colon \mathbb{R} \to \mathbb{R}$ are differentiable, then for x such that $g(x) \neq 0$,
$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}.$$

(g) $\frac{d}{dx}(\sin(x)) = \cos(x)$, $\frac{d}{dx}(\cos(x)) = -\sin(x)$, $\frac{d}{dx}(\tan(x)) = (\sec(x))^2$, $\frac{d}{dx}(\sec(x)) = \sec(x)\tan(x)$, $\frac{d}{dx}(\csc(x)) = -\csc(x)\cot(x)$, $\frac{d}{dx}(\cot(x)) = -(\csc(x))^2$.

(h) (**Chain rule**) If $f, g \colon \mathbb{R} \to \mathbb{R}$ are differentiable, then
$$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x).$$

Exercise 3.1. Prove (a), (c), (d) and (g).

Remark 3.2. In Chapter 3, we will learn how to prove (b) for general $n \in \mathbb{R}$. For now, we can only prove (b) for $n \in \mathbb{Z}$, $n > 0$.

Proof of (b), assuming n is a positive integer. Let n be a positive integer. Let $x, h \in \mathbb{R}$. From the Binomial Theorem, $(x + h)^n = \sum_{j=0}^{n} x^j h^{n-j} \frac{n!}{j!(n-j)!}$. So, we have

$$\lim_{h \to 0} \frac{(x+h)^n - x^n}{h} = \lim_{h \to 0} \frac{\left(\sum_{j=0}^{n} x^j h^{n-j} \frac{n!}{j!(n-j)!}\right) - x^n}{h} = \lim_{h \to 0} \left(\sum_{j=0}^{n-1} x^j h^{(n-1)-j} \frac{n!}{j!(n-j)!}\right).$$

For $j = 0, 1, \ldots, n-1$, each term $\lim_{h \to 0} x^j h^{(n-1)-j}$ exists, and for $j = 0, 1, \ldots, n-2$, this limit is zero. Therefore, the limit of the sum is the sum of the limits, i.e. we have

$$\lim_{h \to 0} \frac{(x+h)^n - x^n}{h} = \sum_{j=0}^{n-1} \lim_{h \to 0} \left(x^j h^{(n-1)-j} \frac{n!}{j!(n-j)!}\right) = x^{n-1} \frac{n!}{(n-1)!} = nx^{n-1}.$$

$\square$

Proof of (e). Let $x, h \in \mathbb{R}$. Then
$$\frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$
$$= \frac{f(x+h)g(x+h) - f(x)g(x+h) + g(x+h)f(x) - f(x)g(x)}{h}$$
$$= \frac{f(x+h)g(x+h) - f(x)g(x+h)}{h} + \frac{g(x+h)f(x) - f(x)g(x)}{h}$$
$$= g(x+h)\frac{f(x+h) - f(x)}{h} + f(x)\frac{g(x+h) - g(x)}{h}.$$

Since g is differentiable, g is continuous by Proposition 1.2, so that $\lim_{h \to 0} g(x+h) = g(x)$. Applying our limit laws (which ones, and how are they justified?), we get

$$\lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$
$$= \left(\lim_{h \to 0} g(x+h)\frac{f(x+h) - f(x)}{h}\right) + \left(\lim_{h \to 0} f(x)\frac{g(x+h) - g(x)}{h}\right)$$
$$= g(x)f'(x) + f(x)g'(x).$$

$\square$

Proof of (f). Let x such that $g(x) \neq 0$. We first show that $(1/g(x))$ is differentiable, and

$$\frac{d}{dx}\left[\frac{1}{g(x)}\right] = -\frac{g'(x)}{(g(x))^2}. \qquad (*)$$

Observe,

$$\frac{1}{h}\left(\frac{1}{g(x+h)} - \frac{1}{g(x)}\right) = \frac{1}{h}\left(\frac{g(x)}{g(x)g(x+h)} - \frac{g(x+h)}{g(x)g(x+h)}\right)$$
$$= \frac{g(x) - g(x+h)}{h}\frac{1}{g(x)g(x+h)}. \qquad (**)$$

Since g is differentiable, g is continuous by Proposition 1.2, so $\lim_{h\to 0} g(x+h) = g(x)$. Using our quotient limit law,

$$\lim_{h\to 0}\frac{1}{g(x+h)} = \frac{1}{\lim_{h\to 0} g(x+h)} = \frac{1}{g(x)}.$$

So, taking the limit of $(**)$, and using our limit law for products (why does this apply?),

$$\lim_{h\to 0}\frac{1}{h}\left(\frac{1}{g(x+h)} - \frac{1}{g(x)}\right) = \left(\lim_{h\to 0}\frac{g(x) - g(x+h)}{h}\right)\left(\lim_{h\to 0}\frac{1}{g(x)g(x+h)}\right) = \frac{g'(x)}{(g(x))^2}.$$

Since $\lim_{h\to 0}\frac{1}{h}\left(\frac{1}{g(x+h)} - \frac{1}{g(x)}\right)$ exists, $1/g(x)$ is differentiable, and formula $(*)$ is proven. We can now conclude (f) from the product rule (e). Observe

$$\frac{d}{dx}\left(f(x)\frac{1}{g(x)}\right) = f(x)\frac{d}{dx}\left(\frac{1}{g(x)}\right) + f'(x)\frac{1}{g(x)}$$
$$= -f(x)\frac{g'(x)}{(g(x))^2} + f'(x)\frac{g(x)}{(g(x))^2} = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}.$$

$\square$

Proof of (g), for sin(x). We first recall that $\lim_{h\to 0}\frac{\sin(h)}{h} = 1$. Also, $\lim_{h\to 0}\frac{\cos(h)-1}{h} = 0$. To compute the latter limit, note that

$$\frac{\cos(h)-1}{h} = \frac{\cos(h)-1}{h}\frac{\cos(h)+1}{\cos(h)+1} = -\frac{(\sin(h))^2}{h(\cos(h)+1)} = -\frac{\sin(h)}{h}\frac{\sin(h)}{\cos(h)+1}.$$

Therefore, applying our limit rule for products and quotients,

$$\lim_{h\to 0}\frac{\cos(h)-1}{h} = -\left(\lim_{h\to 0}\frac{\sin(h)}{h}\right)\left(\lim_{h\to 0}\frac{\sin(h)}{\cos(h)+1}\right) = -1\cdot 0 = 0.$$

We can now prove (g). Recall that $\sin(x+h) = \sin(x)\cos(h) + \cos(x)\sin(h)$. So, applying our limit law for addition (why can we apply it?), we get

$$\lim_{h\to 0}\frac{\sin(x+h) - \sin(x)}{h} = \lim_{h\to 0}\frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h}$$
$$= \sin(x)\left(\lim_{h\to 0}\frac{(\cos(h)-1)}{h}\right) + \cos(x)\left(\lim_{h\to 0}\frac{\sin(h)}{h}\right)$$
$$= 0\cdot\sin(x) + \cos(x) = \cos(x).$$

$\square$

Proof of (h). Let $h, k \in \mathbb{R}$ with $h \neq 0$. Let $y = g(x)$. Let $u(h) = g(x + h) - g(x) - hg'(x)$, and let $v(k) = f(y + k) - f(k) - kf'(y)$. Since f and g are differentiable,

$$\lim_{h \to 0} \frac{u(h)}{h} = -g'(x) + \lim_{h \to 0} \frac{g(x + h) - g(x)}{h} = 0,$$

$$\lim_{k \to 0} \frac{v(k)}{k} = -f'(x) + \lim_{k \to 0} \frac{f(y + k) - f(y)}{k} = 0.$$

So, there exists functions $\varepsilon(h)$ and $\eta(k)$ such that $\varepsilon(h) \to 0$ as $h \to 0$, $\eta(k) \to 0$ as $k \to 0$, $|u(h)| = |h|\, \varepsilon(h)$, and $|v(k)| = |k|\, \eta(k)$. Let $k = g(x + h) - g(x)$. Then

$$|k| = |hg'(x) + u(h)| \leq |h|\, (|g'(x)| + \varepsilon(h)). \qquad (*)$$

Also, applying our definitions,

$$f(g(x + h)) - f(g(x)) - hf'(g(x))g'(x) = f(y + k) - f(y) - hf'(g(x))g'(x)$$
$$= v(k) + kf'(y) - hf'(y)g'(x) = (k - hg'(x))f'(y) + v(k)$$
$$= u(h)f'(y) + v(k).$$

So, applying the triangle inequality, the definitions of $\varepsilon(h)$ and $\eta(k)$, and then $(*)$,

$$\left| \frac{f(g(x + h)) - f(g(x)) - hf'(g(x))g'(x)}{h} \right| \leq |f'(y)| \frac{|u(h)|}{|h|} + \frac{|v(k)|}{|h|}$$

$$\leq |f'(y)|\, \varepsilon(h) + \eta(k) \frac{|k|}{|h|}$$

$$\leq |f'(y)|\, \varepsilon(h) + \eta(k)(|g'(x)| + \varepsilon(h)).$$

Define, $F(h) = 0$,

$$G(h) = \left| \frac{f(g(x + h)) - f(g(x)) - hf'(g(x))g'(x)}{h} \right|,$$

$$H(h) = |f'(y)|\, \varepsilon(h) + \eta(k)(|g'(x)| + \varepsilon(h)).$$

From $(*)$, $k \to 0$ as $h \to 0$. Therefore, $\eta(k) \to 0$ as $h \to 0$. Since $\varepsilon(h) \to 0$ as $h \to 0$ as well, we conclude that $H(h) \to 0$ as $h \to 0$. Since $F(h) \leq G(h) \leq H(h)$, and $\lim_{h \to 0} F(h) = \lim_{h \to 0} H(h) = 0$, we conclude from the Squeeze Theorem that $\lim_{h \to 0} G(h) = 0$. That is,

$$0 = \lim_{h \to 0} \left| \frac{f(g(x + h)) - f(g(x)) - hf'(g(x))g'(x)}{h} \right|$$

$$= \lim_{h \to 0} \left| -f'(g(x))g'(x) + \frac{f(g(x + h)) - f(g(x))}{h} \right|.$$

$\square$

4. Selected Exercises

Exercise 4.1. (Ordering products for a store) (Thomas' Calculus, p. 171) One of the formulas for inventory management says that the average weekly cost of ordering, paying for, and holding merchandise is

$$A(q) = \frac{km}{q} + cm + \frac{hq}{2},$$

where q is the quantity you order when things run low (shoes, radios, brooms, or whatever the item might be); k is the cost of placing an order (the cost is the same, no matter how often you make an order); c is the cost of one item (a constant); m is the number of items sold each week (a constant); and h is the weekly holding cost per item (a constant that takes into account things such as space, utilities, insurance, and security). Find dA/dq and d^2A/dq^2, and interpret your results in terms of the constants.

Exercise 4.2. (A Population Growth Model) Let $P(t)$ denote the number of people living on Earth at time t, where t is measured in years after 1900. So, $P(0)$ is the population of Earth in 1900. It is estimated that the maximum population of Earth is around 10^{10} people. One can make a simplified model of the population of Earth by asserting that $P(0) = (1.7) \times 10^9$, and that $P(t)$ satisfies the following equation for $t > 0$.

$$\frac{dP}{dt}(t) = (.022)P(t)\left(1 - \frac{P(t)}{10^{10}}\right) \qquad (*)$$

In words, the rate of change of the population is proportional to the current population, multiplied by the fraction of "vacant space." Intuitively, the more people there are, the faster they can reproduce. However, as the maximum population of 10^{10} is reached, the rate of change of the population should be very low, since there are less resources and less free space. The constant .022 is just an arbitrary constant that affects the growth rate of P. This kind of model can also model the growth of bacteria.

   (a) Verify that the following function satisfies equation $(*)$, and $P(0) = (1.7) \times 10^9$.

$$P(t) = \frac{1}{(1.7 \times 10^9)^{-1}e^{-.022t} + 10^{-10}(1 - e^{-.022t})}.$$

     (Use the formula $\frac{d}{dx}e^{\alpha x} = \alpha e^{\alpha x}$, $\alpha \in \mathbb{R}$.) Later on, we will learn how to derive $P(t)$ directly from $(*)$.

   (b) What is $\lim_{t \to \infty} P(t)$? What is $\lim_{t \to -\infty} P(t)$?

   (c) Using (a), plot dP/dt. For what t is dP/dt the largest? At this t, what is $P(t)$?

   (d) In the later part of the twentieth century, many people became worried that the world population would explode uncontrollably. If we trust our model of population growth, then does part (c) explain this worry?

Exercise 4.3. (Free Fall and Terminal Velocity) Suppose we have an object of mass m (in kilograms) that is dropped from a stationary position at time $t = 0$. Let $v(t)$ denote the vertical velocity of the object at time t. We use the convention that negative velocity denotes falling toward the ground. The force of gravity is about $-10m$. The force due to air friction is approximately proportional to $v(t)$. Specifically, there is a constant $b > 0$ (depending on the shape of the body, the material properties of its surface, etc.) such that the force is $bv(t)$. From Newton's second law, force is equal to mass multiplied by acceleration. Since acceleration is the derivative of velocity, we have

$$m\frac{dv}{dt}(t) = -10m - bv(t) \qquad (*)$$

Recall also that $v(0) = 0$.

(a) Verify that the following function satisfies equation $(*)$, and $v(0) = 0$.

$$v(t) = \frac{10m}{b}(e^{-bt/m} - 1).$$

As in the previous exercise, we will learn how to find this equation directly from $(*)$.

(b) What is $\lim_{t \to \infty} v(t)$? This limit is known as the terminal velocity.

Exercise 4.4. For the following functions, determine whether or not $f'(0)$ exists. If it exists, compute its value.

(a) $\quad f(x) = \begin{cases} x \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$,
$\qquad$
(b) $\quad f(x) = \begin{cases} x^2 \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$.

Exercise 4.5. Find the equation of the tangent to the curve at the given point.

(a) $\quad y = 4(\sin(x))^2$, $(x, y) = (\pi/6, 1)$,
$\qquad$
(b) $\quad y = \dfrac{x^2 - 1}{x^2 + 1}$, $(x, y) = (0, -1)$.

Exercise 4.6. Find the equation of the tangent line and normal line to the curve at the given point.

(a) $\quad y = \sqrt{1 + 4\sin(x)}$, $(x, y) = (0, 1)$,
$\qquad$
(b) $\quad x^2 + 4xy + y^2 = 13$, $(x, y) = (2, 1)$.

Exercise 4.7. (**Cardiac output**, Thomas' Calculus, p. 220) In the late 1860s, Adolf Fick, a professor of physiology in the Faculty of Medicine in Würzberg, Germany, developed one of the methods we use today for measuring how much blood your heart pumps in a minute. Your cardiac output as you read this sentence is probably about 7 L/min. At rest it is likely to be a bit under 6 L/min. If you are a trained marathon runner running a marathon, your cardiac output can be as high as 30 L/min.

Your cardiac output can be calculated with the formula

$$y = \frac{Q}{D},$$

where Q is the number of mLs of CO_2 you exhale in a minute, and D is the difference between the CO_2 concentration (mL/L) in the blood pumped to the lungs and the CO_2 concentration in the blood returning from the lungs. With $Q = 233$ mL/min and $D = 97 - 56 = 41$ mL/L,

$$y = \frac{233\text{mL/min}}{41\text{mL/L}} \approx 5.68\text{L/min},$$

fairly close to the 6 L/min that most people have at basal (resting) conditions. (Data courtesy of J. Kenneth Herd, M.D., Quillan College of Medicine, East Tennessee State University.)

Suppose that when $Q = 233$ and $D = 41$, we also know that D is decreasing at the rate of 2 units a minute but that Q remains unchanged. What is happening to the cardiac output?

5. SELECTED PROBLEMS

Problem 5.1. A lighthouse sits 1 mile from the shore. Let P be the point on the shore that is closest to the lighthouse. The light completes four revolutions per minute, at a constant speed. How fast is the light moving along the shore, when the light is 1 mile above the point P?

Problem 5.2. (**Making Coffee**, Thomas' Calculus, p. 220) Coffee is draining from a conical filter into a cylindrical coffeepot at a rate of 10 in^3/min. The filter is a circular cone with a height of 6 inches and a 6 inch diameter. The cylindrical coffeepot has a diameter of 6 inches.

(a) How fast is the level in the pot rising when the coffee in the cone is 5 inches deep?
(b) How fast is the level in the cone falling at this point in time?

Problem 5.3. Let $f\colon \mathbb{R} \to \mathbb{R}$ be differentiable. Show that

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{f(x) - f(x-h)}{h}$$

Problem 5.4. (MIT 18.01SC, Exam 1.3) A particle is moving along a vertical axis so that its position (in meters) at time t (in seconds) is given by the equation

$$y(t) = t^3 - 3t + 3, \quad t \geq 0.$$

Determine the total distance traveled by the particle in the first three seconds.

Problem 5.5. Can you come up with a function that is everywhere continuous, but nowhere differentiable? (This is an extremely difficult question.) At very least, try to come up with some intuitive ideas about how to make such a function.

This question is not just an academic one. Stock prices and microscopic particles can be modeled by Brownian motion. Brownian motion can be thought of as a function $f\colon [0, \infty) \to \mathbb{R}$ with $f(0) = 0$. This function satisfies certain randomness properties. These properties make f continuous but nowhere differentiable. However, f satisfies a property that is in between differentiability and continuity.

Let $0 < \alpha < 1$. We say that $f\colon \mathbb{R} \to \mathbb{R}$ is Hölder continuous of order α if: for all $x, y \in \mathbb{R}$,

$$|f(x) - f(y)| \leq |x - y|^\alpha$$

It turns out that Brownian paths are Hölder continuous of order α for any $0 < \alpha < 1/2$.

(a) Let $0 < \alpha < 1$. Show that, if f is Hölder continuous of order α, then f is continuous.
(b) Give an example of a function that is Hölder continuous of order $1/2$, but which is not differentiable at some point in its domain.

COURANT INSTITUTE, NEW YORK UNIVERSITY, NEW YORK NY 10012
E-mail address: heilman@cims.nyu.edu