4: OPTIMIZATION

STEVEN HEILMAN

1. Optimization and Derivatives

Nothing takes place in the world whose meaning is not that of some maximum or minimum.
Leonhard Euler

At this stage, Euler's statement may seem to exaggerate, but perhaps the Exercises in Section 4 and the Problems in Section 5 may reinforce his views. These problems and exercises show that many physical phenomena can be explained by maximizing or minimizing some function. We therefore begin by discussing how to find the maxima and minima of a function. In Section 2, we will see that the first and second derivatives of a function play a crucial role in identifying maxima and minima, and also in drawing functions. Finally, in Section 3, we will briefly describe a way to find the zeros of a general function. This procedure is known as Newton's Method. As we already see in Algorithm 1.2(1) below, finding the zeros of a general function is crucial within optimization.

We now begin our discussion of optimization. We first recall the Extreme Value Theorem from the last set of notes. In Algorithm 1.2, we will then describe a general procedure for optimizing a function.

Theorem 1.1. *(**Extreme Value Theorem**) Let $a < b$. Let $f: [a,b] \to \mathbb{R}$ be a continuous function. Then f achieves its minimum and maximum values. More specifically, there exist $c, d \in [a,b]$ such that: for all $x \in [a,b]$, $f(c) \leq f(x) \leq f(d)$.*

Algorithm 1.2. A procedure for finding the extreme values of a differentiable function $f: [a,b] \to \mathbb{R}$.

(1) Find $x \in (a,b)$ with $f'(x) = 0$.
(2) Compare the values of $f(a), f(b)$ and f evaluated at $x \in (a,b)$ with $f'(x) = 0$.
(3) Choose the largest and smallest values of f from part (2).

We now prove that step (3) of Algorithm 1.2 finds the extreme values of f on $[a,b]$.

Proof. This proof is similar to the proof of Rolle's Theorem. By the Extreme Value Theorem, let $x_0, x_1 \in [a,b]$ such that $f(x_0) \leq f(x) \leq f(x_1)$ for all $x \in [a,b]$. We want to show that x_0, x_1 were found in Step (2). Since the function f was evaluated at a and b in Step (2), it remains to check the case that $x_0, x_1 \in (a,b)$.

If $f = 0$ everywhere there is nothing to prove, so we assume otherwise. That is, assume $f(x_1) > f(x_0)$. Let $h > 0$ so that h is less than the minimum of $|x_1 - a|/2$, $|x_1 - b|/2$,

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$|x_0 - a|/2$, and $|x_0 - b|/2$. By our choice of x_1, we have $f(x_1 + h) \leq f(x_1)$ and $f(x_1 - h) \leq f(x_1)$. Therefore, by the Squeeze Theorem,

$$0 \geq \lim_{h \to 0^+} \frac{f(x_1 + h) - f(x_1)}{h} = f'(x_1) = \lim_{h \to 0^+} \frac{f(x_1) - f(x_1 - h)}{h} \geq 0.$$

We conclude that $f'(x_1) = 0$, so x_1 as found in Step (1) of Algorithm 1.2. Similarly, by our choice of x_0, we have $f(x_0 + h) \geq f(x_0)$ and $f(x_0 - h) \geq f(x_0)$. So, by the Squeeze Theorem,

$$0 \geq \lim_{h \to 0^+} \frac{f(x_0) - f(x_0 - h)}{h} = f'(x_0) = \lim_{h \to 0^+} \frac{f(x_0 + h) - f(x_0)}{h} \geq 0.$$

So, $f'(x_0) = 0$, and x_0 as found in Step (1) of Algorithm 1.2. So, in any case, we have found the extreme values of f on $[a, b]$ in part (3) of Algorithm 1.2. □

Remark 1.3. Algorithm 1.2 can be remembered concisely as follows. First, check the interior for $f' = 0$. Then check the boundary for f. Here the interior refers to (a, b) and the boundary refers to the individual points a, b. It is very important to check the boundary. For example, consider $f(x) = x$ on the interval $[0, 1]$. Then $f'(x) = 1$, so f has no critical numbers on $(0, 1)$. And f has its minimum value at $x = 0$ and its maximum value at $x = 1$.

Remark 1.4. For a differentiable function $f: [a, b] \to \mathbb{R}$, if $f'(x) = 0$ for $x \in (a, b)$, then x is not necessarily a maximum or minimum value of f on $[a, b]$. Consider for example $f(x) = x^3$ on $[-1, 1]$. Then $f'(x) = 3x^2$, and $f'(x) = 0$ for $x = 0$. However, $f(0) = 0$, $f(1) = 1$ and $f(-1) = -1$, so $x = 0$ is not an extreme value of f on $[-1, 1]$.

Remark 1.5. If we only know that the function f is continuous, then we can change Algorithm 1.2 by replacing Step (1) with the following

(1') Find $x \in (a, b)$ such that $f'(x) = 0$, or such that $f'(x)$ does not exist.

For example, consider $f(x) = |x|$ on $[-1, 1]$. The minimum value of f on $[-1, 1]$ occurs at $x = 0$. However, $f'(0)$ does not exist.

Definition 1.6. Let $f: (a, b) \to \mathbb{R}$ be a continuous function. We say that $x \in (a, b)$ is a **critical number** of f if one of the two following things occurs: $f'(x) = 0$, or $f'(x)$ does not exist.

Definition 1.7. Let $f: (a, b) \to \mathbb{R}$ be a continuous function. We say that $x \in (a, b)$ is a **local maximum** of f if there is a $\delta(x) > 0$ such that: if $y \in (a, b)$ satisfies $|y - x| < \delta(x)$, then $f(x) \geq f(y)$. We say that $x \in (a, b)$ is a **local minimum** of f if there is a $\delta(x) > 0$ such that: if $y \in (a, b)$ satisfies $|y - x| < \delta(x)$, then $f(x) \leq f(y)$.

2. Shapes of Curves/ Derivative Tests

By examining the first and second derivatives of a function, we can say a lot about the properties of that function. In the previous section, we saw that the zeros of the first derivative tell us most of the information about the maximum and minimum values of a given function. To see what else the derivatives tell us, see the MIT JAVA Applet, graph features. Below, we will describe several tests on the first and second derivatives that allow us to find several properties of a function, as in this Applet.

Proposition 2.1. (**Constant Test**) Let $f: (a, b) \to \mathbb{R}$ be a differentiable function. If $f'(x) = 0$ for all $x \in (a, b)$, then f is a constant function.

Proposition 2.2. Let $f, g\colon (a,b) \to \mathbb{R}$ be differentiable functions. If $f'(x) - g'(x) = 0$ for all $x \in (a,b)$, then there is a $c \in \mathbb{R}$ such that $f - g = c$.

Proposition 2.3. (**Increasing/Decreasing Test**) Let $f\colon (a,b) \to \mathbb{R}$ be differentiable.

(1) If $f'(x) > 0$ for all $x \in (a,b)$, and if $x, y \in (a,b)$ with $x < y$, then $f(x) < f(y)$.
(2) If $f'(x) < 0$ for all $x \in (a,b)$, and if $x, y \in (a,b)$ with $x < y$, then $f(x) > f(y)$.
(3) If $f'(x) \geq 0$ for all $x \in (a,b)$, and if $x, y \in (a,b)$ with $x < y$, then $f(x) \leq f(y)$.
(4) If $f'(x) \leq 0$ for all $x \in (a,b)$, and if $x, y \in (a,b)$ with $x < y$, then $f(x) \geq f(y)$.

Proposition 2.4. (**First Derivative Test**) Let $c \in (a,b)$ be a critical number for a continuous function $f\colon (a,b) \to \mathbb{R}$. Assume that f is differentiable on (a,c) and on (c,b).

(1) If $f'(x) > 0$ on (a,c), and if $f'(x) < 0$ on (c,b), then f has a local maximum at $x = c$.
(2) If $f'(x) < 0$ on (a,c), and if $f'(x) > 0$ on (c,b), then f has a local minimum at $x = c$.
(3) If $f'(x) > 0$ on $(a,c) \cup (c,b)$, or if $f'(x) < 0$ on $(a,c) \cup (c,b)$, then f does not have a local maximum or a local minimum at $x = c$.

Proposition 2.5. (**Concavity Test**) Let $f\colon (a,b) \to \mathbb{R}$ be a twice differentiable function.

(1) If $f''(x) > 0$ on (a,b), then f is concave up. That is, if $y(x) = ax + b$ denotes a tangent line to f, then $f(x) \geq y(x)$ for all $x \in (a,b)$.
(2) If $f''(x) < 0$ on (a,b), then f is concave down. That is, if $y(x) = ax + b$ denotes a tangent line to f, then $f(x) \geq y(x)$ for all $x \in (a,b)$.

Proposition 2.6. (**Second Derivative Test**) Let $f\colon (a,b) \to \mathbb{R}$ be a twice differentiable function. Let $c \in (a,b)$. Assume that $f'(c)$ and $f''(c)$ exist. Assume also that $f'(c) = 0$, and that f'' is continuous near c.

(1) If $f''(c) > 0$, then f has a local minimum at c.
(2) If $f''(c) < 0$, then f has a local maximum at c.

Remark 2.7. To remember the difference between concave up and concave down, consider $f(x) = x^2$ and $g(x) = -x^2$. Then $f''(x) = 2$, and f lies above its tangent lines. And $g''(x) = -2$, and g lies below its tangent lines. These two examples are also good to keep in mind for the Second Derivative Test.

Our analysis of the derivative tests is mostly based on the Mean Value Theorem, which we recall from the previous set of notes.

Theorem 2.8. ***(Mean Value Theorem)*** *Let $f\colon [a,b] \to \mathbb{R}$ be continuous function that is differentiable on (a,b). Then there exists c with $c \in (a,b)$ and*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Proof of the Constant Test, Proposition 2.1. Let $x, y \in (a,b)$ with $x < y$. From the Mean Value Theorem, Theorem 2.8, there exists $c \in (x,y)$ such that $f'(c) = (f(y) - f(x))/(y - x)$. Since $f'(c) = 0$, we must therefore have $f(x) = f(y)$, so that f is constant □

Proof of Proposition 2.2. Apply Proposition 2.1 to $f - g$. □

Proof of the Increasing/Decreasing Test, Proposition 2.3. (1) Let $x, y \in (a,b)$ with $x < y$. From the Mean Value Theorem, Theorem 2.8, there exists $c \in (x,y)$ such that $f'(c) = (f(y) - f(x))/(y - x)$. Since $f'(c) > 0$, we must therefore have $f(x) < f(y)$.

(2) Let $x, y \in (a, b)$ with $x < y$. From the Mean Value Theorem, Theorem 2.8, there exists $c \in (x, y)$ such that $f'(c) = (f(y) - f(x))/(y - x)$. Since $f'(c) < 0$, we must therefore have $f(x) > f(y)$.

(3) Let $x, y \in (a, b)$ with $x < y$. From the Mean Value Theorem, Theorem 2.8, there exists $c \in (x, y)$ such that $f'(c) = (f(y) - f(x))/(y - x)$. Since $f'(c) \geq 0$, we must therefore have $f(x) \leq f(y)$.

(4) Let $x, y \in (a, b)$ with $x < y$. From the Mean Value Theorem, Theorem 2.8, there exists $c \in (x, y)$ such that $f'(c) = (f(y) - f(x))/(y - x)$. Since $f'(c) \leq 0$, we must therefore have $f(x) \geq f(y)$. □

Proof of the First Derivative Test, Proposition 2.4. (1) Apply Algorithm 1.2 to find $x \in [a, b]$ such that f achieves its maximum at x. From this algorithm, we know that $x = a$, or $x = b$, or x is a critical number. By our assumptions, $x = c$ is the only critical number of f. So, it remains to show that $f(c) > f(a)$ and $f(c) > f(b)$. The assertion $f(c) > f(a)$ follows from the Increasing Test, Proposition 2.3(1). The assertion $f(c) > f(b)$ follows from the Decreasing Test, Proposition 2.3(2).

(2) Apply Algorithm 1.2 to find $x \in [a, b]$ such that f achieves its minimum at x. From this algorithm, we know that $x = a$, or $x = b$, or x is a critical number. By our assumptions, $x = c$ is the only critical number of f. So, it remains to show that $f(c) < f(a)$ and $f(c) < f(b)$. The assertion $f(c) < f(a)$ follows from the Decreasing Test, Proposition 2.3(2). The assertion $f(c) < f(b)$ follows from the Increasing Test, Proposition 2.3(1).

(3) Assume that $f' > 0$ on $(a, c) \cup (c, b)$. Let $x \in (a, c)$ and let $y \in (c, b)$. We claim that $f(x) < f(c) < f(y)$. Therefore, c is a not a local maximum or a local minimum. The assertion $f(x) < f(c)$ follows from the Increasing Test, Proposition 2.3(1). The assertion $f(c) < f(y)$ also follows from the Increasing Test, Proposition 2.3(1).

We now consider the case $f' < 0$ on $(a, c) \cup (c, b)$. Let $x \in (a, c)$ and let $y \in (c, b)$. We claim that $f(x) > f(c) > f(y)$. Therefore, c is a not a local maximum or a local minimum. The assertion $f(x) > f(c)$ follows from the Decreasing Test, Proposition 2.3(2). The assertion $f(c) > f(y)$ also follows from the Decreasing Test, Proposition 2.3(2). □

Proof of the Concavity Test, Proposition 2.5. (1) Fix $x \in (a, b)$ and let $x < y < b$. Since $f''(x) > 0$, the Increasing Test, Proposition 2.3(1), shows that $f'(x)$ is increasing on (a, b). By the Mean Value Theorem, Theorem 2.8, there exists $c \in (x, y)$ such that $f'(c) = (f(y) - f(x))/(y - x)$. Since f' is increasing on (a, b), $f'(c) > f'(x)$, so $(f(y) - f(x))/(y - x) > f'(x)$, so $f(y) - f(x) > f'(x)(y - x)$, so $f(y) > f'(x)(y - x) + f(x)$. The left side of this inequality is $f(y)$, and the right side of this inequality is the tangent line of f at x. So, f is concave up on (x, b). Since $x \in (a, b)$ is arbitrary, f is concave up on (a, b).

(2) This proof is essentially identical to that of (1). □

Proof of the Second Derivative Test, Proposition 2.6. (1) Since f'' is continuous near c, the definition of continuity says that there exists a, b such that $c \in (a, b)$ and such that $f''(x) > 0$ for all $x \in (a, b)$. Then the Concavity Test, Proposition 2.5, says that $f(c+h) \geq f'(c)h + f(c)$ for $c + h \in (a, b)$. Since $f'(c) = 0$, we have $f(c + h) \geq f(c)$, so c is a local minimum of f.

(2) This proof is essentially identical to that of (1).

□

The connection between the second derivative and the shape of the function may not be entirely clear. Perhaps the following proposition will make this connection more concrete.

Proposition 2.9. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a twice differentiable function. Then

$$f''(x) = \lim_{h\to 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2}$$

In words, $f''(x)$ measures the change between $f(x)$ and its two "neighbors."

Proof. From Problem 1 of the third set of notes,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{f(x) - f(x-h)}{h}. \qquad (*)$$

Let $g(x) = (f(x+h) - f(x))/h$. Then

$$\begin{aligned} \frac{g(x) - g(x-h')}{h'} &= \frac{\frac{f(x+h)-f(x)}{h} - \frac{f(x+h-h')-f(x-h')}{h}}{h'} \\ &= \frac{f(x+h) + f(x-h') - f(x) - f(x+h-h')}{hh'} \end{aligned}$$

Fix x and let $\varepsilon > 0$. By the definition of the limit, there exists $\delta_1(\varepsilon) > 0$ such that:

$$\text{if } 0 < |h'| < \delta_1(\varepsilon), \text{ then } \left| \frac{g(x) - g(x-h')}{h'} - g'(x) \right| < \varepsilon/2.$$

Note that $g'(x) = (f'(x+h) - f'(x))/h$. So there exists $\delta_2(\varepsilon) > 0$ such that:

$$\text{if } 0 < |h| < \delta_2(\varepsilon), \text{ then } |g'(x) - f''(x)| < \varepsilon/2.$$

Let $\delta(\varepsilon)$ be the minimum of $\delta_1(\varepsilon)$ and $\delta_2(\varepsilon)$. Let $h = h'$ with $0 < |h| < \delta(\varepsilon)$. Then

$$\begin{aligned} \left| \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} - f''(x) \right| &= \left| \frac{g(x) - g(x-h)}{h} - f''(x) \right| \\ &= \left| \frac{g(x) - g(x-h)}{h} - g'(x) + g'(x) - f''(x) \right| \\ &\le \left| \frac{g(x) - g(x-h)}{h} - g'(x) \right| + |g'(x) - f''(x)| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, letting $h \to 0$ shows that $(*)$ holds. ∎

3. Newton's Method

From Algorithm 1.2, we see that most of the work of optimizing a function involves finding x such that $f'(x) = 0$. In practice, the function $f'(x)$ is sometimes too complicated, and the equation $f'(x) = 0$ may be difficult to solve for x. Thankfully, Newton came up with a general method for finding the zeros of general differentiable functions. This procedure does not work all the time, but it works enough of the time that it is quite useful. It is used by your calculator, for example, whenever you try to find the zeros of a function.

Algorithm 3.1. Newon's Method, a general way to find the roots of a differentiable function $f\colon \mathbb{R} \to \mathbb{R}$.

(1) Choose any point $x_0 \in \mathbb{R}$.
(2) Compute the tangent line of f at x_0: $y(x) = f'(x_0)(x - x_0) + f(x_0)$.

(3) Find x_1 such that $y(x_1) = 0$. This is the intersection of the tangent line $y(x)$ with the x-axis. Note that x_1 satisfies

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

(4) Return to step (2), but replace x_0 with x_1. At the n^{th} iteration of the algorithm, compute the tangent line of f at x_n in step (2), and then find an x_{n+1} in step (3) which is a zero of the tangent line. So, in general we iterate the following equation.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Remark 3.2. To see an illustration of Newton's Method, see the UMN Applet, Newton Example 1. In many examples, it only takes a few iterations of the algorithm to get a good approximation for a zero of f.

Remark 3.3. If we ever find a point in step (2) where $f' = 0$, the algorithm will be unable to continue. Actually, if we repeatedly encounter points where f' is close to zero, then Newton's Method will not work very well. For an illustration, see the UMN Applet, Newton Example 2. There are a few ways to adjust the function f or the starting value x_0 to deal with these issues, so that a suitable modification of Algorithm 3.1 can find the zeros of many functions.

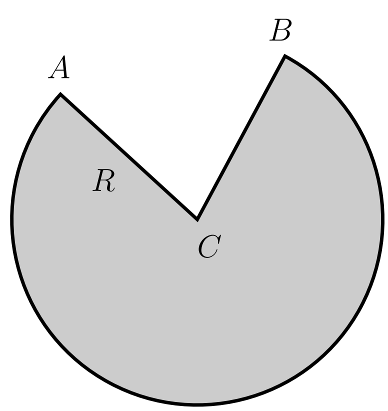
4. Selected Exercises from the Textbook

Exercise 4.1. Find two numbers whose difference is 100 and whose product is a minimum.

Exercise 4.2. Suppose 1200 cm^2 of material is available to make a box with a square base and an open top. Find the largest possible volume of the box.

Exercise 4.3. Find the point on the line $y = 2x + 3$ that is closest to the origin.

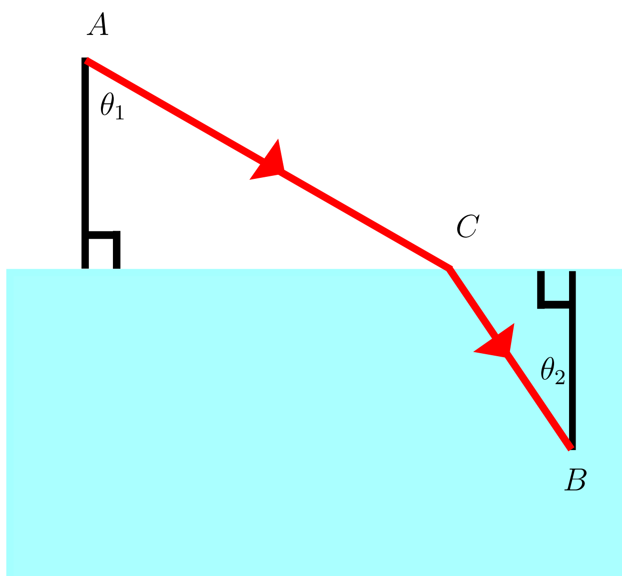
Exercise 4.4. Suppose we have a circular piece of paper with center C and radius R. Let A, B be two points on the boundary of the circle. A cone-shaped paper drinking cup is made from the circular piece of paper by cutting out a sector CAB and joining the edges CA and CB. Find the maximum capacity of such a cup.



Exercise 4.5. Let v_1 be the velocity of light in air and let v_2 be the velocity of light in water. Suppose a light ray travels from point A in the air to a point B in the water. Let C be the point where the light ray first hits the water. At this point, the light bends, and then continues in a straight line. According to Fermat's Principle, the ray of light has traveled along the path ACB that minimizes the elapsed travel time between the points A and B. Let θ_1 be the angle of incidence and let θ_2 be the angle of refraction. From Fermat's Principle, derive Snell's Law:

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_1}{v_2}$$

As the light travels from A, θ_1 is the angle that the ray makes with a vertical line. Also, as the light hits B, θ_2 is the angle that the ray makes with a vertical line.



Exercise 4.6. For a fish swimming at a speed v relative to the water, the energy expenditure per unit of time is proportional to v^3. It is believed that migrating fish try to minimize the total energy required to swim a fixed distance. If the fish are swimming against a current of speed u ($u < v$), then the time required to swim a distance L is $L/(v-u)$, and the total energy E required to swim the distance L is given by

$$E(v) = av^3 \frac{L}{v-u}$$

Here a is an arbitrary constant.

(a) Determine the value of v that minimizes E.
(b) Sketch the graph of E.

Note: This result has been verified experimentally. Migrating fish swim against a current at a speed 50% greater than the speed of the current.

Problem 5.1. (**Branching angles for blood vessels and pipes**, Thomas' Calculus, p. 552) When a smaller pipe branches off from a larger one in a flow system, we may want it to run off at an angle that is best from some energy-saving point of view. We might require, for instance, that energy loss due to friction can be minimized along the section AOB shown in Figure 1. In this problem, the pipe AC is fixed, and the point B is fixed. The point O is variable. So, we know that the larger pipe AC branches at O into the smaller pipe OB. We just allow the point O to vary. A law due to Poiseuille states that the loss of energy due to friction in nonturbulent fluid flow is proportional to the length of the path. This energy loss is also inversely proportional to the fourth power of the radius of the pipe. Let k be a constant, let d_1 be the length of AO, let d_2 be the length of OB, let θ be the angle BOC, let R be the radius of the pipe AC, and let r be the radius of the pipe OB. Then the energy loss along AO is $(kd_1)/R^4$, and the energy loss along OB is $(kd_2)/r^4$. We are required to minimize the energy loss along the path AOB. So, we need to minimize

$$L = k\frac{d_1}{R^4} + k\frac{d_2}{r^4}$$

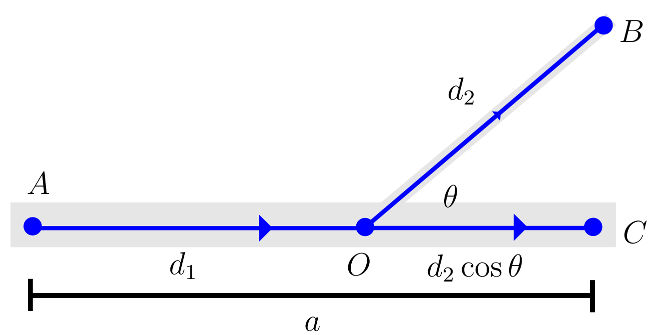


Figure 1.

To review, $AC = a$ and $BC = b$ are fixed. By the definitions of our constants, we have

$$d_1 + d_2 \cos\theta = a, \qquad \text{and} \qquad d_2 \sin\theta = b.$$

So, $d_2 = b/\sin\theta$ and $d_1 = a - d_2\cos\theta = a - b\cos\theta/\sin\theta$. We can therefore express L as a function of θ as follows

$$L = L(\theta) = k\left(\frac{a - b\frac{\cos\theta}{\sin\theta}}{R^4} + \frac{b}{r^4 \sin\theta}\right).$$

(a) Show that the critical number of θ for which $dL/d\theta = 0$ is given by

$$\theta_c = \cos^{-1}\left(\frac{r^4}{R^4}\right).$$

(b) If the ratio of the pipe radii is $r/R = 5/6$, estimate (in degrees) the optimal branching angle given in part (a).

The analysis used here can also be used to explain the angles at which arteries branch in an animal's body.

Problem 5.2. Show that a degree 3 polynomial has at most three real roots. What can you say about the degree n case?

Problem 5.3. (**Minimizing Costs**, Thomas' p. 284) A cabinetmaker uses mahogany to produce 5 furnishings each day. Each delivery of one container of wood costs \$5000, and storage of that material is \$10 per day per unit stored, where a unit is the amount of material needed by her to produce 1 furnishing. How much material should be ordered each time and how often should the material be delivered to minimize her average daily cost in the production cycle between deliveries?

Problem 5.4. (**Ordering products for a store, continued**) (Thomas' Calculus, p. 290) One of the formulas for inventory management says that the average weekly cost of ordering, paying for, and holding merchandise is

$$A(q) = \frac{km}{q} + cm + \frac{hq}{2},$$

where q is the quantity you order when things run low (shoes, radios, brooms, or whatever the item might be); k is the cost of placing an order (the cost is the same, no matter how often you make an order); c is the cost of one item (a constant); m is the number of items sold each week (a constant); and h is the weekly holding cost per item (a constant that takes into account things such as space, utilities, insurance, and security).

(a) Your job, as the inventory manager for your store, is to find the quantity that will minimize $A(q)$. What is it?

(b) Shipping costs sometimes depend on order size. When they do, it is more realistic to replace k by $k + bq$, the sum of k and a constant multiple of q. What is the most economical quantity to order now?

Problem 5.5. (**How we cough**, Thomas' Calculus, p. 290) When we cough, the trachea (windpipe) contracts to increase the velocity of the air going out. This raises the question of how much it should contract to maximize the velocity of air, and whether the trachea really contracts that much when we cough.

Let r_0 be the rest radius of the trachea in centimeters, and let c be a positive constant whose value depends in part on the length of the trachea. Under reasonable assumptions about how the air near the wall is slowed by friction, the average air flow velocity v can be modeled by the equation

$$v = c(r_0 - r)r^2 \text{ cm/sec}, \qquad \frac{r_0}{2} \leq r \leq r_0.$$

Show that v is greatest when $r = (2/3)r_0$. That is, the velocity is greatest when the trachea is about 33% contracted. The remarkable fact is that X-ray photographs confirm that the trachea contracts about this much during a cough.

Problem 5.6. (**An inequality for positive integers**, Thomas' Calculus p. 290) Let a, b, c, d be positive integers. Show that

$$\frac{(a^2+1)(b^2+1)(c^2+1)(d^2+1)}{abcd} \geq 16.$$

Problem 5.7. (Challenge problem) Create an optimization problem where the optimum does not exist.

Problem 5.8. (Challenge problem) Suppose a farmer has 100 feet of fence, and she wants to maximize the area that is enclosed by the fence. What shape should she build the fence? If you can come up with the best shape, can you prove that it is the best? If you cannot prove that it is the best, can you at least give an intuitive argument that your shape is the best?

This question has ancient roots and many applications. The Greeks knew the answer, but a rigorous proof was not known until the 1800s.

Courant Institute, New York University, New York NY 10012
E-mail address: heilman@cims.nyu.edu