3: THE SHANNON SAMPLING THEOREM

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1. Introduction

In this final set of notes, we would like to end where we began. In the first set of notes, we mentioned that mathematics could be used to show how CDs and MP3s can reproduce our music well, even though the file sizes of MP3s are relatively small. In this set of notes, we attempt to describe how this is possible. Surprisingly, the exponential function will play a key role. We assume familiarity with differentiation and integration theory.

We begin our discussion with Fourier series. In the class, we have familiarized ourselves with infinite series. Some of the most interesting infinite series are associated to functions $f\colon [0,1] \to \mathbb{R}$. In particular, we can encode a lot of information of a function $f\colon [0,1] \to \mathbb{R}$ via a certain infinite series, which is called the Fourier series of f. It is this series that will lead us to an analysis of CDs and MP3s.

2. Review for the Final Exam

Before the discussion of Fourier series, I would like to review the types of proofs that we have seen in this course. I would also like to review some ways of deciding which proof we want to use. Let P, Q, R, S, T be statements, let A be a set, for $a \in A$ let $P(a)$ be a statement, and for $n \in \mathbb{N}$, let $P(n)$ be a statement.

Type of proof	Typical Appearance	Typical proof method
Direct Proof	$P \Rightarrow Q$	Find e.g. R, S, T, show $P \Rightarrow R \Rightarrow S \Rightarrow T \Rightarrow Q$.
Proof by cases	$\forall\ a \in A$, prove $P(a)$. There are too many different things to do for a direct proof.	There are a few disjoint subsets of A, e.g. A_1, A_2, A_3 such that $A_1 \cup A_2 \cup A_3 = A$, and it seems easier to prove $P(a)$ separately for $a \in A_1$, $a \in A_2$, $a \in A_3$.
Proof by contrapositive	$P \Rightarrow Q$ seems hard, so prove $\sim Q \Rightarrow \sim P$.	Do a direct proof of $\sim Q \Rightarrow \sim P$.
Proof by contradiction	Prove P. A direct proof of P seems hard.	Assume that $\sim P$ holds, derive some contradiction. Due to the contradiction, P must hold.
Disproof by counterexample	$\sim(\forall\ a \in A,\ P(a))$, i.e. $\exists\ a \in A, \sim P(a)$	Find $a \in A$ such that $\sim P(a)$ holds.
Inductive proof	$\forall\ n \in \mathbb{N}$, $P(n)$ holds	Prove $P(1)$. Prove: $\forall\ n \in \mathbb{N}$, $P(n) \Rightarrow P(n+1)$. Therefore, $P(1) \Rightarrow P(2) \Rightarrow P(3) \Rightarrow \cdots$.

Table 1. Different Proof Strategies

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Exercise 2.1. Using the homeworks and the practice final, review particular examples where the above strategies are used.

3. Primer: Fourier analysis, Poisson Summation

Let $f\colon [0,1] \to \mathbb{R}$ be a real valued function such that $f(0) = f(1)$. Below, we always assume that f is a smooth function. That is, $\forall\ n \in \mathbb{N}$, the n^{th} derivative of f, $d^n f(x)/dx^n$ exists and is continuous, and $\exists\ C_n \in \mathbb{R}, C_n < \infty$ with $|d^n f(x)/dx^n| \leq C_n$, $\forall\ x \in [0,1]$. Here C_n is allowed to depend on f. The smoothness of f will eliminate some technicalities in our presentation. Occasionally, we will not mention when the smoothness of f is used. As an exercise, try to find where the smoothness of f is used. Let $n \in \mathbb{Z}$, and define a sequence $\ldots, a_{-2}, a_{-1}, a_0, a_1, \ldots$ by the following formula

$$a_n = \int_0^1 f(x)e^{-2\pi i n x}dx = \left(\int_0^1 f(x)\cos(-2\pi n x)dx\right) + i\left(\int_0^1 f(x)\sin(-2\pi n x)dx\right)$$

Define the **Fourier series** of f to be

$$\sum_{n\in\mathbb{Z}} a_n e^{2\pi i n x}$$

The following theorem shows that a smooth function is equal to its Fourier series. In other words, we can recover any value of f from a specific infinite sequence of numbers. Here we can already see that these results may be useful for storing information about an audio file. Suppose $f\colon [0,1] \to \mathbb{R}$ represents a sound wave. That is, for $x \in [0,1]$, $f(x)$ is equal to the amplitude of the sound wave at time x. It seems difficult to store a function f, but it seems easier to store a sequence of numbers. Ideally, we would only store a finite sequence of numbers, and then recover the function f reasonably well. This strategy can actually be carried out, and it will be described in more detail in Section 6.

The following theorem is relatively easy to prove, since we have assumed that f is a smooth function. However, if f is not smooth, many issues arise.

Theorem 3.1. *Let $f\colon [0,1] \to \mathbb{R}$ be smooth, with $f(0) = f(1)$. Define a_n as above. Then*

$$f(x) = \sum_{n\in\mathbb{Z}} a_n e^{2\pi i n x}$$

Proof. Since $a_n = \int_0^1 f(x)e^{-2\pi i n x}dx$ and $f(0) = f(1)$, we can integrate by parts twice to get

$$a_n = \int_0^1 f(x)\frac{d}{dx}\left(\frac{1}{-2\pi i n}e^{-2\pi i n x}\right)dx = \frac{1}{2\pi i n}\int_0^1 f'(x)e^{-2\pi i n x}dx$$

$$a_n = \frac{1}{(2\pi i n)^2}\int_0^1 f''(x)e^{-2\pi i n x}dx$$

Recall that $|e^{-2\pi i n x}| = 1$. So, taking absolute values of both sides and then moving the absolute values inside the integral gives

$$|a_n| \leq \frac{1}{4\pi^2 n^2}\int_0^1 |f''(x)|\,dx \leq \frac{1}{n^2}\frac{C_2}{4\pi^2} \qquad (*)$$

For $N \in \mathbb{N}$, the definition of a_n shows that

$$\sum_{n=-N}^{N} a_n e^{2\pi inx} = \sum_{n=-N}^{N} \left(\int_0^1 f(y) e^{-2\pi iny} dy \right) e^{2\pi inx} = \int_0^1 f(y) \left(\sum_{n=-N}^{N} e^{2\pi in(x-y)} \right) dy \qquad (**)$$

Let $x \in (0,1)$. Using the formula for summation of geometric series, the identity $\sin(x) = (e^{ix} - e^{-ix})/2i$, and the formula $\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$, $a, b \in \mathbb{R}$,

$$\begin{aligned}
\sum_{n=-N}^{N} e^{2\pi inx} &= \sum_{n=0}^{N} (e^{2\pi ix})^n + \sum_{n=1}^{N} (e^{-2\pi ix})^n \\
&= \frac{1 - e^{2\pi i(N+1)x}}{1 - e^{2\pi ix}} + \frac{e^{-2\pi iNx} - 1}{1 - e^{2\pi ix}} \\
&= \frac{e^{-2\pi iNx} - e^{2\pi i(N+1)x}}{1 - e^{2\pi ix}} \cdot \frac{e^{-\pi ix}}{e^{-\pi ix}} = \frac{e^{-2\pi i(N+1/2)x} - e^{2\pi i(N+1/2)x}}{e^{-\pi ix} - e^{\pi ix}} \\
&= \frac{e^{2\pi i(N+1/2)x} - e^{-2\pi i(N+1/2)x}}{2i} \frac{2i}{e^{\pi ix} - e^{-\pi ix}} = \frac{\sin(\pi(2N+1)x)}{\sin(\pi x)} \\
&= \frac{\sin(2\pi Nx)\cos(\pi x)}{\sin \pi x} + \cos(2\pi Nx) \qquad (\dagger)
\end{aligned}$$

Let $N \in \mathbb{N}$. For $n \neq 0, n \in \mathbb{N}$, $\int_0^1 e^{2\pi inx} dx = \frac{1}{2\pi in}[e^{2\pi inx}]_{x=0}^{x=1} = \frac{1}{2\pi in}[1-1] = 0$. So, $\int_0^1 \sum_{n=-N}^{N} e^{2\pi inx} dx = \int_0^1 dx = 1$. Fix $x \in (0,1)$. For $y \in [0,1]$, $y \neq x$, let $F(y) = (f(x) - f(x-y))/y$, and for $y = x$, let $F(y) = f'(x)$. Since f is smooth, F is also smooth. Applying the equality $\int_0^1 \sum_{n=-N}^{N} e^{2\pi inx} dx = 1$ along with $(**)$,

$$\begin{aligned}
f(x) - \sum_{n=-N}^{N} a_n e^{2\pi inx} &= f(x) \int_0^1 \sum_{n=-N}^{N} e^{2\pi iny} dy - \int_0^1 f(y) \left(\sum_{n=-N}^{N} e^{2\pi in(x-y)} \right) dy \\
&= f(x) \int_0^1 \sum_{n=-N}^{N} e^{2\pi iny} dy - \int_0^1 f(x-y) \left(\sum_{n=-N}^{N} e^{2\pi iny} \right) dy \qquad \text{, changing variables} \\
&= \int_0^1 [f(x) - f(x-y)] \sum_{n=-N}^{N} e^{2\pi iny} \cdot \frac{y}{y} dy \\
&= \int_0^1 F(y) \frac{y}{\sin(\pi y)} \cos(\pi y) \sin(2\pi Ny) + \int_0^1 F(y) y \cos(2\pi Ny) dy \qquad \text{, by } (\dagger)
\end{aligned}$$

Now, $F(y)\frac{y}{\sin(\pi y)}\cos(\pi y)$ is a smooth function, and $F(y)y$ is a smooth function. So, integrating by parts twice as in $(*)$, there exists $C > 0$ such that

$$\left| f(x) - \sum_{n=-N}^{N} a_n e^{2\pi inx} \right| \leq \frac{C}{N^2}$$

Letting $N \to \infty$ completes the Theorem. $\square$

Let $f\colon \mathbb{R} \to \mathbb{R}$ be a real valued function. Throughout these notes, we assume that f is a smooth, rapidly decaying function. That is, $\forall\ n, j \in \mathbb{N}$, $\exists\ C_{n,j} \in \mathbb{R}$ such that the n^{th} derivative of f, $d^n f(x)/dx^n$, exists and is continuous, and

$$\forall\, x \in \mathbb{R}, \qquad \left|\frac{d^n f(x)}{dx^n}\right| |x|^j \leq C_{n,j}$$

Here $C_{n,j}$ is allowed to depend on the function f. This strong assumption on f is not really needed, but it eliminates some technicalities in the presentation below.

Let $\xi \in \mathbb{R}$. Given $f\colon \mathbb{R} \to \mathbb{R}$, we define the **Fourier Transform** of f by the formula

$$\widehat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i x\xi} dx = \left(\int_{\mathbb{R}} f(x)\cos(-2\pi x\xi)dx\right) + i\left(\int_{\mathbb{R}} f(x)\sin(-2\pi x\xi)dx\right)$$

To get a handle on this object, we consider the following example.

Example 3.2. For $x \in \mathbb{R}, x \in [-1/2, 1/2]$, let $h(x) = 1$. For $x \in \mathbb{R}, x \notin [-1/2, 1/2]$, let $h(x) = 0$. We compute $\widehat{h}(\xi)$.

$$\begin{aligned}\widehat{h}(\xi) &= \int_{\mathbb{R}} h(x)e^{-2\pi i x\xi}dx = \int_{-1/2}^{1/2} e^{-2\pi i x\xi}dx = \frac{1}{-2\pi i\xi}[e^{-2\pi i x\xi}]_{x=-1/2}^{x=1/2} \\ &= \frac{e^{\pi i\xi} - e^{-\pi i\xi}}{2\pi i\xi} = \frac{\sin(\pi\xi)}{\pi\xi}\end{aligned}$$

So, h is just a flat function on the interval $[-1/2, 1/2]$, and $\widehat{h}$ is a wavy function that somewhat resembles the function $1/|\xi|$, for large ξ.

The following two theorems connect f and $\widehat{f}$. Below, we will think of f as our sound wave, and $\widehat{f}$ will contain the frequency information of the sound wave f. In order to store the sound wave f in an audio file, it is useful to pass back and forth between the amplitude function f, and the frequency function $\widehat{f}$. Therefore, being able to relate both f and $\widehat{f}$ will be very useful, especially in Section 5.

Theorem 3.3. ***(Poisson Summation Formula)*** *Let $x \in \mathbb{R}$. Then*

$$\sum_{n\in\mathbb{Z}} f(x+n) = \sum_{n\in\mathbb{Z}} \widehat{f}(n)e^{2\pi i n x}$$

Proof. Let $k \in \mathbb{Z}$. Note that $e^{2\pi i k} = (\cos(2\pi k)) + i(\sin 2\pi k) = 1$. For $x \in [0,1]$, let $F(x) = \sum_{n\in\mathbb{Z}} f(x+n)$. Then $F\colon [0,1] \to \mathbb{R}$, $F(0) = F(1)$, and F is smooth. Observe

$$\begin{aligned}\int_0^1 F(x)e^{-2\pi i n x}dx &= \int_0^1 \sum_{n\in\mathbb{Z}} f(x+n)e^{-2\pi i n x}dx = \sum_{n\in\mathbb{Z}}\int_0^1 f(x+n)e^{-2\pi i n x}dx \\ &= \sum_{n\in\mathbb{Z}}\int_n^{n+1} f(x)e^{-2\pi i n(x-n)}dx \qquad \text{, changing variables} \\ &= \sum_{n\in\mathbb{Z}}\int_n^{n+1} f(x)e^{-2\pi i n x}e^{2\pi i n^2}dx = \sum_{n\in\mathbb{Z}}\int_n^{n+1} f(x)e^{-2\pi i n x}dx \\ &= \int_{\mathbb{R}} f(x)e^{-2\pi i n x}dx = \widehat{f}(n)\end{aligned}$$

Therefore, applying Theorem 3.1 to F,

$$\sum_{n\in\mathbb{Z}} f(x+n) = F(x) = \sum_{n\in\mathbb{Z}} \left(\int_0^1 F(y)e^{-2\pi i n y} dy \right) e^{2\pi i n x} = \sum_{n\in\mathbb{Z}} \widehat{f}(n) e^{2\pi i n x}$$

□

The following proof skips some details. Try to find out where these details are skipped. These details can be filled in, but they require techniques beyond this course.

Theorem 3.4. ***(Fourier inversion)***

$$f(x) = \int_{\mathbb{R}} \widehat{f}(\xi) e^{2\pi i x \xi} d\xi$$

Proof. Let $\varepsilon > 0$, and let $g(x) = (1/\varepsilon)e^{-\pi x^2/\varepsilon^2}$. One can compute that $\widehat{g}(\xi) = e^{-\pi\varepsilon^2 x^2}$, and $\widehat{\widehat{g}}(x) = g(x)$. Also, for $h(x) = \widehat{g}(x)$ one can show that $\int_{\mathbb{R}} f(x)\widehat{h}(x)dx = \int_{\mathbb{R}} \widehat{f}(x)h(x)dx$. That is, $\int_{\mathbb{R}} f(x)g(x) = \int_{\mathbb{R}} \widehat{f}(x)\widehat{g}(x)$, so

$$\int_{\mathbb{R}} f(x)\frac{1}{\varepsilon}e^{-\pi x^2/\varepsilon^2} dx = \int_{\mathbb{R}} \widehat{f}(\xi) e^{-\pi\varepsilon^2\xi^2} d\xi \qquad (*)$$

Letting ε go to zero, the right side of $(*)$ becomes $\int_{\mathbb{R}} \widehat{f}(\xi)d\xi$. As ε goes to zero, $\frac{1}{\varepsilon}e^{-\pi\xi^2/\varepsilon^2}$ becomes more and more localized around the point $\xi = 0$, so the left side of $(*)$ becomes $f(0)$ as ε goes to zero. That is,

$$f(0) = \int_{\mathbb{R}} \widehat{f}(\xi) d\xi \qquad (**)$$

Finally, for $x \in \mathbb{R}$ fixed, apply $(**)$ with $u(y) = f(x+y)$ in place of $f(x)$. By changing variables, note that $\widehat{u}(\xi) = \int_{\mathbb{R}} f(x+y)e^{-2\pi i y\xi}dy = \int_{\mathbb{R}} f(y)e^{-2\pi i (y-x)\xi}dx = e^{2\pi i x\xi}\widehat{f}(\xi)$. Therefore, $(**)$ applied to u says

$$f(x) = u(0) = \int_{\mathbb{R}} \widehat{u}(\xi)d\xi = \int_{\mathbb{R}} \widehat{f}(\xi)e^{2\pi i x\xi}d\xi$$

□

4. An Aside: The Heisenberg Uncertainty Principle

In this section, f and g are smooth rapidly decaying functions, $f, g\colon \mathbb{R} \to \mathbb{R}$.

Theorem 4.1. ***(Cauchy-Schwarz)***

$$\int_{\mathbb{R}} f(x)g(x)dx \leq \left(\int_{\mathbb{R}} |f(x)|^2 dx \right)^{1/2} \left(\int_{\mathbb{R}} |g(x)|^2 dx \right)^{1/2}$$

Proof. Let $\delta = -(\int_{\mathbb{R}} f(x)g(x)dx)/(\int_{\mathbb{R}} |g(x)|^2\, dx)$. Then

$$
\begin{aligned}
0 \le \int_{\mathbb{R}} (f(y) + \delta g(y))^2 dy &= \int_{\mathbb{R}} |f(y)|^2\, dy + \delta^2 \int_{\mathbb{R}} |g(y)|^2\, dy + 2\delta \int_{\mathbb{R}} f(y)g(y)dy \\
&= \int_{\mathbb{R}} |f(y)|^2\, dy + \frac{(\int_{\mathbb{R}} f(x)g(x)dx)^2}{\int_{\mathbb{R}} |g(x)|^2\, dx} - 2\frac{(\int_{\mathbb{R}} f(x)g(x)dx)^2}{\int_{\mathbb{R}} |g(x)|^2\, dx} \\
&= \int_{\mathbb{R}} |f(y)|^2\, dy - \frac{(\int_{\mathbb{R}} f(x)g(x)dx)^2}{\int_{\mathbb{R}} |g(x)|^2\, dx}
\end{aligned}
$$

Therefore, $(\int_{\mathbb{R}} f(x)g(x)dx)^2 \le (\int_{\mathbb{R}} f(x)dx)(\int_{\mathbb{R}} g(x)dx)$, as desired. □

Consider the following binary operation on functions $f, g\colon \mathbb{R} \to \mathbb{R}$

$$(f * g)(x) = \int_{\mathbb{R}} f(x-y)g(y)dy$$

Exercise 4.2. Show that $f*g$ is a commutative binary operation, i.e. show $f*g(x) = g*f(x)$.

Theorem 4.3.

$$\widehat{f*g}(\xi) = \widehat{f}(\xi)\widehat{g}(\xi)$$

Proof.

$$
\begin{aligned}
\widehat{f*g}(\xi) &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x-y)g(y)dy \right) e^{-2\pi i x\xi} dx = \int_{\mathbb{R}} g(y) \left(\int_{\mathbb{R}} f(x-y)e^{-2\pi i x\xi} dx \right) dy \\
&= \int_{\mathbb{R}} g(y) \left(\int_{\mathbb{R}} f(x)e^{-2\pi i (x+y)\xi} dx \right) dy = \int_{\mathbb{R}} g(y)e^{-2\pi i y\xi} dy \int_{\mathbb{R}} f(x)e^{-2\pi i x\xi} dx = \widehat{g}(\xi)\widehat{f}(\xi)
\end{aligned}
$$

□

Theorem 4.4. ***(Plancherel)*** $\int_{\mathbb{R}} |f(x)|^2\, dx = \int_{\mathbb{R}} |\widehat{f}(\xi)|^2 d\xi$

Proof. Let $g(x) = \overline{f(-x)}$. Then $\widehat{g}(\xi) = \int_{\mathbb{R}} \overline{f(-x)} e^{-2\pi i x\xi} dx = \overline{\int_{\mathbb{R}} f(x) e^{-2\pi i x\xi} dx} = \overline{\widehat{f}(\xi)}$. Let $h(x) = f * g(x)$. By Theorem 4.3 and the previous sentence, $\widehat{h}(\xi) = \widehat{f}(\xi)\widehat{g}(\xi) = \widehat{f}(\xi)\overline{\widehat{f}(\xi)} = |\widehat{f}(\xi)|^2$. Now, applying Theorem 3.4 to h,

$$
\begin{aligned}
\int_{\mathbb{R}} |\widehat{f}(\xi)|^2 d\xi = \int_{\mathbb{R}} \widehat{h}(\xi) d\xi = h(0) &= \int_{\mathbb{R}} f(x)g(0-x)dx \qquad \text{, by Exercise 4.2} \\
&= \int_{\mathbb{R}} f(x)\overline{f(x)}dx = \int_{\mathbb{R}} |f(x)|^2\, dx
\end{aligned}
$$

□

Theorem 4.5. ***(Heisenberg Uncertainty Principle)*** *Suppose* $\int_{\mathbb{R}} |f(x)|^2\, dx = 1$. *Then*

$$\left(\int_{\mathbb{R}} x^2 |f(x)|^2\, dx \right) \left(\int_{\mathbb{R}} \xi^2 |\widehat{f}(\xi)|^2 d\xi \right) \ge \frac{1}{16\pi^2}$$

Proof. From Fourier inversion, $f(x) = \int_{\mathbb{R}} \widehat{f}(\xi) e^{-2\pi i x\xi} d\xi$, so $f'(x) = \int_{\mathbb{R}} (-2\pi i\xi)\widehat{f}(\xi)e^{-2\pi i x\xi} d\xi$, and using Fourier inversion again, $\widehat{(f')}(\xi) = (-2\pi i \xi)\widehat{f}(\xi)$. So, by Theorem 4.4,

$$\int_{\mathbb{R}} |f'(x)|^2\, dx = \int_{\mathbb{R}} |\widehat{f'(\xi)}|^2 d\xi = 4\pi^2 \int_{\mathbb{R}} \xi^2 |\widehat{f}(\xi)|^2 d\xi \qquad (*)$$

We now begin with the assumption $\int_{\mathbb{R}} |f(x)|^2 \, dx = 1$ and integrate by parts.

$$1 = \int_{\mathbb{R}} |f(x)|^2 \, dx = -\int_{\mathbb{R}} x \frac{d}{dx} |f(x)|^2 \, dx = -\int_{\mathbb{R}} x \frac{d}{dx} (f(x)\overline{f(x)}) dx$$
$$= -\int_{\mathbb{R}} (x f'(x)\overline{f(x)} + x\overline{f'(x)} f(x)) dx$$

Taking absolute values inside the integral, then applying Theorem 4.1 and $(*)$,

$$1 \leq 2 \int_{\mathbb{R}} |x| \, |f(x)| \, |f'(x)| \, dx \leq 2 \left(\int_{\mathbb{R}} x^2 \, |f(x)|^2 \, dx \right)^{1/2} \left(\int_{\mathbb{R}} |f'(x)|^2 \, dx \right)^{1/2}$$
$$= 4\pi \left(\int_{\mathbb{R}} x^2 \, |f(x)|^2 \, dx \right)^{1/2} \left(\int_{\mathbb{R}} \xi^2 |\widehat{f}(\xi)|^2 d\xi \right)^{1/2}$$

□

5. The Shannon Sampling Theorem, or how CDs and MP3s work

Theorem 5.1. *Let $A \in \mathbb{R}$, $A > 0$. Suppose $\widehat{f}(\xi) = 0$ for $\{\xi \in \mathbb{R} \colon \; |\xi| < A\}$. Then $f(x)$ can be recovered from its values at the following discrete sets of points, $\{y \in \mathbb{R} \colon \, y = n/2A, n \in \mathbb{Z}\}$. More specifically, f can be written in the following way*

$$f(x) = \sum_{n \in \mathbb{Z}} f\left(\frac{n}{2A}\right) \frac{\sin(\pi(2Ax - n))}{\pi(2Ax - n)}$$

For simplicity of presentation, we will prove the theorem only in the case $A = 1/2$. We leave it as an exercise to adjust this proof to apply to the more general case.

Theorem 5.2. *Suppose $\widehat{f}(\xi) = 0$ for $\{\xi \in \mathbb{R} \colon \; |\xi| < 1/2\}$. Then $f(x)$ can be recovered from its values at the integers. More specifically, f can be written in the following way*

$$f(x) = \sum_{n \in \mathbb{Z}} f(n) \frac{\sin(\pi(x - n))}{\pi(x - n)}$$

Proof. Let $g(x) = \widehat{f}(x)$. From Theorem 3.4, $\widehat{f}(\xi) = g(\xi) = \int_{\mathbb{R}} \widehat{g}(x) e^{2\pi i x \xi} dx$. Since $\widehat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i x \xi} dx$, we conclude that $\int_{\mathbb{R}} [\widehat{g}(x) - f(-x)] e^{2\pi i x \xi} = 0$. From Theorem 3.4 again, we conclude that $\widehat{g}(x) = f(-x)$. Now, if $\xi \in \mathbb{R}$, the given information says $\sum_{n \in \mathbb{Z}} \widehat{f}(\xi + n) = \widehat{f}(\xi)$. So, applying Theorem 3.3 to $g(x) = \widehat{f}(x)$ and using $\widehat{g}(x) = f(-x)$,

$$\widehat{f}(\xi) = \sum_{n \in \mathbb{Z}} \widehat{f}(\xi + n) = \sum_{n \in \mathbb{Z}} f(-n) e^{2\pi i n \xi} = \sum_{n \in \mathbb{Z}} f(n) e^{-2\pi i n \xi}$$

Define h as in Example 3.2. Then $\widehat{f}(\xi) = h(\xi) \sum_{n \in \mathbb{Z}} f(n) e^{-2\pi i n \xi}$, so by Theorem 3.4 and Example 3.2,

$$f(x) = \int_{\mathbb{R}} \widehat{f}(\xi) e^{2\pi i x \xi} d\xi = \sum_{n \in \mathbb{Z}} f(n) \int_{\mathbb{R}} h(\xi) e^{2\pi i \xi (x - n)} = \sum_{n \in \mathbb{Z}} f(n) \frac{\sin(\pi(x - n))}{\pi(x - n)}$$

□

6. Conclusion

We can now finally describe how CDs and MP3s work. Suppose $f : \mathbb{R} \to \mathbb{R}$ represents the amplitude of a sound wave. That is, for some time $t \in \mathbb{R}$, $f(t)$ denotes the height of the sound wave f. The Fourier Transform $\widehat{f}(\xi)$ represents the frequency of the sound wave f. That is, $\widehat{f}(\xi)$ is large if f is oscillating at the frequency ξ. Humans only hear sounds at frequencies less than about $20,000$ Hertz. So, we may conservatively assume that $\widehat{f}(\xi) = 0$ on the set $\{\xi \in \mathbb{R} : |\xi| > 22,000\}$. Then Theorem 5.1 says that, if we sample the height f around $44,000$ times per second (with $1/44,000$ seconds in between each sample), then we can recover f exactly. In particular, on the CD we can store the samples of the function f at these discrete times. We have just described how CDs store and recover sound information.

Now, MP3s take a much different approach to information storage. For CDs, we store samples of the height of the sound wave itself. For MP3s, we store information about the frequencies of the sound wave. The first step is, we consider the sound wave f in separate blocks, say of width .1 seconds. Fr $x \in \mathbb{R}$, suppose $h(x) = 1$ for $x \in [0, .1]$, and $h(x) = 0$ otherwise. Now, consider the Fourier transform of fh. (MP3s do not exactly use the Fourier transform, but they use something similar.) Recall that the Fourier transform of fh expresses how much of a certain frequency fh has. For typical soundwaves, fh will only have a few frequencies that are large. That is, for typical soundwaves, $\widehat{fh}(\xi)$ will be large for very few values of $\xi \in \mathbb{R}$. So, the MP3 only stores the large frequencies, and it ignores the small ones. (The exact way that we implement this procedure affects the size of the MP3 and the quality of the sound. If we throw out too many frequencies, the sound will be terrible. If we do not throw out many frequencies, the file size will be large.) It turns out that, since we store most of the information about $\widehat{fh}$, we can apply Fourier inversion to the frequencies that we stored in the MP3, and we will recover fh pretty well. (Once again, the exact way that we threw out certain frequencies will affect how well we can recover fh. Quantifying this procedure uses many of the ideas that we presented above.)

In summary, the CD stores the values of the physical sound wave, and the MP3 stores (most of) the values of the frequencies of the sound wave. Actually, JPEGs use a similar strategy as MP3s to store visual information. And the transmission of cell phone voice signals and YouTube videos also use schemes similar to the one we described for MP3s. The rigorous foundations of sonic and visual compression serve as only a few of the many examples in which mathematical proofs are vital in the sciences and in our everyday lives.

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