Math 167, Spring 2016, UCLA — Instructor: Steven Heilman

Name: ______________________ UCLA ID: ______________ Date: ___________

Signature: ______________________.
(By signing here, I certify that I have taken this test while refraining from cheating.)

# Mid-Term 1

This exam contains 7 pages (including this cover page) and 5 problems. Check to see if any pages are missing. Enter all requested information on the top of this page.

You may *not* use your books, notes, or any calculator on this exam. You are required to show your work on each problem on the exam. The following rules apply:

- You have 50 minutes to complete the exam, starting at the beginning of class.
- **If you use a theorem or proposition from class or the notes or the book you must indicate this** and explain why the theorem may be applied. It is okay to just say, "by some theorem/proposition from class."
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this. Scratch paper appears at the end of the document.

| Problem | Points | Score |
|---|---|---|
| 1 | 15 | |
| 2 | 20 | |
| 3 | 10 | |
| 4 | 15 | |
| 5 | 15 | |
| Total: | 75 | |

Do not write in the table to the right. Good luck![a]

1. Label the following statements as TRUE or FALSE. If the statement is true, **explain your reasoning**. If the statement is false, **provide a counterexample and explain your reasoning**.

   (a) (3 points) In the game of chess, it is known that both players can force at least a draw.

   TRUE FALSE (circle one)

   (b) (3 points) In the game of Chomp, on any starting game board of any size (finite or infinite), the first player has a winning strategy.

   TRUE FALSE (circle one)

   (c) (3 points) Suppose the game of Nim begins with one pile of 9999 chips and one pile of 10000 chips. Then the first player has a winning strategy.

   TRUE FALSE (circle one)

   (d) (3 points) Suppose the game of Nim begins with the game position $(1, 2, 3)$. Then the first player has a winning strategy.

   TRUE FALSE (circle one)

   (e) (3 points) Let $A$ be a real $2 \times 2$ matrix. Then the the von Neumann Minimax Theorem can be written as follows.

$$\max_{a,b\in[0,1]} \min_{c,d\in[0,1]} (a, b)A \begin{pmatrix} c \\ d \end{pmatrix} = \min_{c,d\in[0,1]} \max_{a,b\in[0,1]} (a, b)A \begin{pmatrix} c \\ d \end{pmatrix}.$$

   TRUE FALSE (circle one)

2. (20 points) Describe the optimal strategies for both players for the two-person zero-sum game described by the payoff matrix below. That is, at the optimal strategy, with what probability does player $I$ play $C$, with what probability does player $I$ play $D$, with what probability does player $II$ play $A$, with what probability does player $II$ play $B$?

| Player $I$ | | Player $II$: A | B |
|---|---|---|---|
| | C | 0 | 1 |
| | D | 2 | 0 |

Prove that these strategies are optimal.

3. (10 points) Let $Y$ be a random variable such that: $Y = 1$ with probability $1/2$, $Y = 4$ with probability $1/2$.

   Let $Z$ be a random variable such that: $Z = 2$ with probability $1/2$ and $Z = 3$ with probability $1/2$. Assume that $Z$ and $Y$ are independent. What is the probability that: $Y = 4$ and $Z = 2$? What is the expected value of $Y \cdot Z$?

4. (15 points) Let $K \subseteq \mathbf{R}^2$ be the following set:

$$K = \{(x, y) \in \mathbf{R}^2 \colon x + y \geq 1,\ -x - y \geq -2,\ -x + 2y \geq 0,\ 2x - y \geq 0\}.$$

Prove that $K$ is convex. Then, find a hyperplane which separates $K$ from the origin $(0, 0)$.

5. (15 points) Let $A, B \subseteq \mathbf{R}^2$ with $A \cap B = \emptyset$. We say that $A, B$ can be *separated* if the following property holds. There exists $z \in \mathbf{R}^2$ and there exists $c \in \mathbf{R}$ such that $z^T a < c < z^T b$ for all $a \in A$ and for all $b \in B$. We say that $A, B$ cannot be separated if it does not hold that $A, B$ can be separated.

   Give an example of two closed, convex sets $A, B \subseteq \mathbf{R}^2$ with $A \cap B = \emptyset$, such that $A, B$ cannot be separated. (As usual, you have to justify your answer. Also, all of the required conditions on $A, B$ must be satisfied. Lastly, drawing a picture might be helpful, but it will not constitute a complete answer.)

(Scratch paper)