167 Midterm 1 Spring Quarter Solutions[1]

1. Question 1

TRUE/FALSE

(a) In the game of chess, it is known that both players can force at least a draw.

FALSE. It could be the case that the white player (or the black player) has a winning strategy. We just don't know which of these three cases occurs (though we know one of them occurs).

(b) In the game of Chomp, on any starting game board of any size (finite or infinite), the first player has a winning strategy.

FALSE. On a 1×1 board, the first player loses automatically. Also, on a $2 \times \infty$ board, the second player has a winning strategy (as discussed on the homework).

(c) Suppose the game of Nim begins with one pile of 9999 chips and one pile of 10000 chips. Then the first player has a winning strategy.

TRUE. The first move is to take one chip from the pile of 10000, resulting in the game position $(9999, 9999)$. No matter what the other player does, the first player can then force both piles to have an equal number of chips, resulting in a win for the first player (since eventually the first player encounters a single pile of chips).

(d) Suppose the game of Nim begins with the game position $(1, 2, 3)$. Then the first player has a winning strategy.

FALSE. The nim sum of the piles $1 \oplus 2 \oplus 3$ is 0. So, the second player has a winning strategy, not the first, by Bouton's Theorem.

(e) Let A be a real 2×2 matrix. Then the the von Neumann Minimax Theorem can be written as follows.

$$\max_{a,b\in[0,1]} \min_{c,d\in[0,1]} (a, b)A \begin{pmatrix} c \\ d \end{pmatrix} = \min_{c,d\in[0,1]} \max_{a,b\in[0,1]} (a, b)A \begin{pmatrix} c \\ d \end{pmatrix}.$$

FALSE. The Theorem assumes $a + b = 1$ and $c + d = 1$ additionally, that is:

$$\max_{a\in[0,1]} \min_{c\in[0,1]} (a, 1-a)A \begin{pmatrix} c \\ 1-c \end{pmatrix} = \min_{c,d\in[0,1]} \max_{a,b\in[0,1]} (a, 1-a)A \begin{pmatrix} c \\ 1-c \end{pmatrix}.$$

2. Question 2

Describe the optimal strategies for both players for the two-person zero-sum game described by the payoff matrix. That is, at the optimal strategy, with what probability does player I play C, with what probability does player I play D, with what probability does player II play A, with what probability does player II play B?

Player II

Player I		A	B
	C	0	1
	D	2	0

Prove that these strategies are optimal.

[1] April 18, 2016,

Solution. Let P denote the payoff matrix. By the Minimax Theorem and by definition of optimal strategy, the optimal strategies are vectors achieving $\max_{x\in\Delta_2}\min_{y\in\Delta_2} x^T Py = \min_{y\in\Delta_2}\max_{x\in\Delta_2} x^T Py$. For example, the function $x\mapsto \min_{y\in\Delta_2} x^T Py$ achieves its maximum at an optimal strategy vector $x\in\Delta_2$. Write $x=(a,1-a)$ and $y=(b,1-b)$ where $a,b\in[0,1]$. Then $x^T Py = (a,1-a)^T(1-b,2b) = a(1-b)+2(1-a)b = a-3ab+2b$. So,

$$\max_{x\in\Delta_2}\min_{y\in\Delta_2} x^T Py = \max_{a\in[0,1]}\min_{b\in[0,1]}(a-3ab+2b).$$

Let $f(a,b)=a-3ab+2b$. Then $\nabla f(a,b)=(1-3b,-3a+2)$. So, if $a>1/3$, we have $\partial f/\partial b<0$. So, if $a>1/3$, $\min_{y\in\Delta_2} x^T Py$ is achieved at $b=1$. And if $a\leq 1/3$, we have $\partial f/\partial b\geq 0$. So, if $a\leq 1/3$, $\min_{y\in\Delta_2} x^T Py$ is achieved at $b=0$. So, for any $a\in[0,1]$, we have $\min_{y\in\Delta_2} x^T Py = \min(-2a+2,a)$. Therefore,

$$\max_{x\in\Delta_2}\min_{y\in\Delta_2} x^T Py = \max_{a\in[0,1]}\min(2(1-a),a) = 2/3.$$

And this maximum is achieved only when $a=2/3$. So, the only optimal strategy for Player I is $x=(2/3,1/3)$. Also, $\partial f/\partial a = 1-3b$. So, if $b>1/3$, $\partial f/\partial a<0$, and $\max_{a\in[0,1]}(a-3ab+2b)$ is achieved at $a=0$. And if $b\leq 1/3$, then $\partial f/\partial a\geq 0$, so $\max_{a\in[0,1]}(a-3ab+2b)$ is achieved at $a=1$. So,

$$\min_{y\in\Delta_2}\max_{x\in\Delta_2} x^T Py = \min_{b\in[0,1]}\max_{a\in[0,1]}(a-3ab+2b) = \min_{b\in[0,1]}\max(2b,1-b) = 2/3.$$

And this minimum is achieved only when $b=1/3$. So, the only optimal strategy for Player II is $y=(1/3,2/3)$. That is, player II will play A with probability $1/3$ and B with probability $2/3$, and player I will play C with probability $2/3$, and D with probability $1/3$.

3. Question 3

Let Y be a random variable such that: $Y=1$ with probability $1/2$, $Y=4$ with probability $1/2$.

Let Z be a random variable such that: $Z=2$ with probability $1/2$ and $Z=3$ with probability $1/2$. Assume that Z and Y are independent. What is the probability that: $Y=4$ and $Z=2$? What is the expected value of $Y\cdot Z$?

Solution. We know $Y=4$ and $Z=2$ with probability equal to: the probability $Y=4$, multiplied by the probability $Z=2$. So, the probability $Y=4$ and $Z=2$ is $(1/2)\cdot(1/2)=1/4$, since Y and Z are independent, so these probabilities multiply.

Using similar reasoning, the expected value of $Y\cdot Z$ is $(1/4)(1\cdot 2)+(1/4)(1\cdot 3)+(1/4)(4\cdot 2)+(1/4)(4\cdot 3) = (1/4)(2+3+8+12) = 25/4$.

4. Question 4

Let $K\subseteq\mathbb{R}^2$ be the following set:

$$K=\{(x,y)\in\mathbb{R}^2\colon x+y\geq 1,\ -x-y\geq -2,\ -x+2y\geq 0,\ 2x-y\geq 0\}.$$

Prove that K is convex. Then, find a hyperplane which separates K from the origin $(0,0)$.

Solution. Let $(a,b),(c,d)\in K$. Let $t\in(0,1)$. We are required to show that $t(a,b)+(1-t)(c,d)\in K$. Since $a+b\geq 1$, $-a-b\geq -2$, $-a+2b\geq 0$, $2a-b\geq 0$, and since $t>0$, we get $t(a+b)\geq t$, $t(-a-b)\geq -2t$, $t(-a+2b)\geq 0$, $t(2a-b)\geq 0$. Similarly, since $c+d\geq 1$, $-c-d\geq -2$, $-c+2d\geq 0$, $2c-d\geq 0$, and since $1-t>0$, we get $(1-t)(c+d)\geq 1-t$,

$(1-t)(-c-d) \geq -2(1-t)$, $(1-t)(-c+2d) \geq 0$, $(1-t)(2c-d) \geq 0$. So, adding up the respective inequalities, we get

$$t(a+b)+(1-t)(c+d) \geq 1, \qquad t(-a-b)+(1-t)(-c-d) \geq -2,$$
$$t(-a+2b)+(1-t)(-c+2d) \geq 0, \qquad t(2a-b)+(1-t)(2c-d) \geq 0.$$

That is, by definition of K, we have $t(a,b)+(1-t)(c,d) \in K$, as desired.

Now, let $z=(1,1) \in \mathbb{R}^2$. Since $x+y \geq 1$ for any $(x,y) \in K$, we have $z^T(x,y) = x+y \geq 1$ for any $(x,y) \in K$. That is, the line $x+y=1/2$ separates K from the origin $(0,0)$, since $z^T(x,y) > 1/2 > 0$ for all $(x,y) \in K$.

5. Question 5

Let $A, B \subseteq \mathbb{R}^2$ with $A \cap B = \emptyset$. We say that A, B can be *separated* if the following property holds. There exists $z \in \mathbb{R}^2$ and there exists $c \in \mathbb{R}$ such that $z^T a < c < z^T b$ for all $a \in A$ and for all $b \in B$. We say that A, B cannot be separated if it does not hold that A, B can be separated.

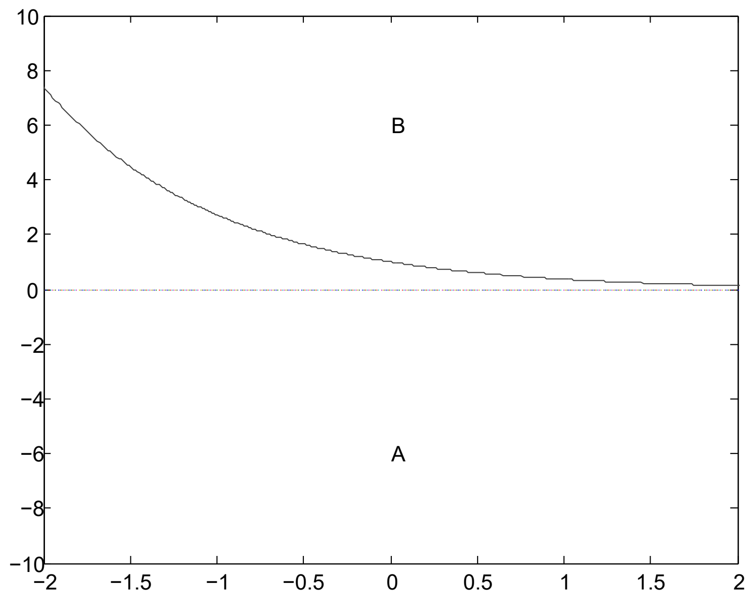
Give an example of two closed, convex sets $A, B \subseteq \mathbb{R}^2$ with $A \cap B = \emptyset$, such that A, B cannot be separated. (As usual, you have to justify your answer. Also, all of the required conditions on A, B must be satisfied. Lastly, drawing a picture might be helpful, but it will not constitute a complete answer.)

Solution. There are several examples that work. Here is one example.

Let $A = \{(x,y) \in \mathbb{R}^2 \colon y \leq 0\}$ and let $B = \{(x,y) \in \mathbb{R}^2 \colon y \geq e^{-x}\}$. Then $A \cap B = \emptyset$ since $y > 0$ whenever $(x,y) \in B$, whereas $y < 0$ whenever $(x,y) \in A$. Now, let $z \in \mathbb{R}^2$. Since $z^T a = z^T b$ holds when $z = 0$, assume that $z \neq 0$. If $x = 0$ then $y \neq 0$, and since $(1,0) \in A$ we have $z^T(1,0) = 0$, and since $(t, e^{-t}) \in B$ for any $t > 0$, we have $z^T(t, e^{-t}) = ye^{-t}$. Letting $t \to \infty$, then $z^T(t, e^{-t})$ decreases to 0. That is, there does not exist a $c \in \mathbb{R}$ such that $z^T a < c < z^T b$ for all $a \in A$ and for all $b \in B$. Now, if $x \neq 0$, then since $(t, 0) \in A$ for any $t \in \mathbb{R}$, we have $z^T(t,0) = xt$. So, as t varies over all $t \in \mathbb{R}$, $z^T(t,0)$ can take any real number value. So, there does not exist $c \in \mathbb{R}$ such that $z^T a < c$ for all $a \in A$. In any case, z does not exist satisfying the condition for A, B being separated.

Lastly, note that A is closed and convex, since it is a closed half plane. Also B is closed since limits preserve nonstrict inequalities (that is, the inequality $y \geq e^{-x}$ is preserved by taking a limit). And B is convex since if $(v,w), (r,u) \in B$, then $w \geq e^{-v}$, $u \geq e^{-r}$, and we are required to show: for any $0 < t < 1$, $(tw + (1-t)u) \geq e^{-(tv+(1-t)r)}$. To prove this inequality, it then suffices to show that $te^{-v} + (1-t)e^{-r} \geq e^{-(tv+(1-t)r)}$. Since the function $x \mapsto e^{-x}$ has strictly positive second derivative for any $x \in \mathbb{R}$, Taylor's Theorem implies that, if $b = tv + (1-t)r$, and if $h(b) = e^{-b}$, $h \colon \mathbb{R} \to \mathbb{R}$, then $h(b+x) \geq h(b) + h'(b)x$. Choosing $x = -tv + tr$ gives $h(r) \geq h(b) + h'(b)t(r-v)$. Choosing $x = -(1-t)r + (1-t)v$ gives $h(v) \geq h(b) + h'(b)(1-t)(v-r)$. Adding these two inequalities, we get the required inequality:

$$te^{-v} + (1-t)e^{-r} \geq h(b) + h'(b)t(1-t)(v-r) + h'(b)t(1-t)(r-v) = h(b).$$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