167 Spring Midterm 2 Solutions[1]

1. Question 1

Recall the prisoner's dilemma, which has the following payoffs.

Prisoner I	Prisoner II: silent	Prisoner II: confess
silent	$(-1,-1)$	$(-10,0)$
confess	$(0,-10)$	$(-8,-8)$

Find all Correlated equilibria for this game.

Solution. Suppose we have a correlated equilibrium z, which is a 2×2 matrix of nonnegative numbers such that $\sum_{i,j=1}^{2} z_{ij} = 1$. The conditions for a correlated equilibrium say the following things:

$$-z_{11} - 10z_{12} \geq -8z_{12}, \qquad -8z_{22} \geq -z_{21} - 10z_{22}.$$

$$-z_{11} - 10z_{21} \geq -8z_{21}, \qquad -8z_{22} \geq -z_{12} - 10z_{22}.$$

Each of the second inequalities always holds, so we only need to find z such that

$$-z_{11} - 10z_{12} \geq -8z_{12}, \qquad -z_{11} - 10z_{21} \geq -8z_{21}.$$

Since $z_{11} \geq 0$, these inequalities imply that

$$-10z_{12} \geq -8z_{12}, \qquad -10z_{21} \geq -8z_{21}.$$

The only nonnegative solution to these inequalities is $z_{12} = z_{21} = 0$. Then the remaining inequalities imply that $-z_{11} \geq 0$, so that $z_{11} = 0$ as well (since $z_{11} \geq 0$). So, $z_{11} = z_{12} = z_{21} = 0$, and consequently $z_{22} = 1$, since $\sum_{i,j=1}^{2} z_{ij} = 1$. So, z defined in the following way is the only Correlated Equilibrium for the Prisoner's Dilemma.

$$z = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = (0,1)^T(0,1).$$

2. Question 2

Find the value of the two-person zero-sum game described by the payoff matrix

$$\begin{pmatrix} 1 & 3 & 3 & 4 \\ 4 & 3 & 3 & 1 \\ 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 4 \end{pmatrix}$$

Solution. The average of the first two rows dominates the third row, so the third row can be ignored in the computation of the value. Similarly, the first row dominates the fourth row. So, we can equivalently compute the value of the matrix

$$\begin{pmatrix} 1 & 3 & 3 & 4 \\ 4 & 3 & 3 & 1 \end{pmatrix}$$

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The middle two columns dominate the average of the first and fourth column. So, the second the third columns can be ignored. That is, we have reduced to computing the value of the matrix

$$\begin{pmatrix} 1 & 4 \\ 4 & 1 \end{pmatrix} = A.$$

Write $x = (s, 1-s)$, $y = (t, 1-t)$, $s, t \in [0,1]$. Then using an Exercise from the notes,

$$\max_{x \in \Delta_2} x^T A y = \max_{i=1,2} (Ay)_i = \max(t + 4(1-t), 4t + (1-t)) = \max(-3t + 4, 3t + 1).$$

And the minimum of this function over all $t \in [0,1]$ occurs when $-3t+4 = 3t+1$, i.e. when $6t = 3$, or $t = 1/2$. So, the value of the game is

$$\min_{y \in \Delta_2} \max_{x \in \Delta_2} x^T A y = \min_{t \in [0,1]} \max(-3t + 4, 3t + 1) = 5/2.$$

3. Question 3

Prove the case $d = 1$ of Sperner's Lemma: Suppose the unit interval $[0,1]$ is partitioned such that $0 = t_0 < t_1 < \cdots < t_n = 1$, where each t_i is marked with a 1 or 2 whenever $0 < i < n$, t_0 is marked 1 and t_n is marked 2. Then the number of ordered pairs (t_i, t_{i+1}), $0 \leq i < n$ with different markings is odd.

Solution. Let $f(t_i)$ be the marking of t_i, for any $0 \leq i < n$. Using a telescoping sum, $1 = 1 - 0 = f(t_n) - f(t_0) = \sum_{i=0}^{n-1} [f(t_{i+1}) - f(t_i)]$. So, we have a sum of elements of $\{-1, 0, 1\}$ which add to 1. The total number of appearances of 1 and -1 must therefore be odd. The total number of appearances of 1 and -1 is equal to the number of ordered pairs (t_i, t_{i+1}), $0 \leq i < n$, with different markings.

4. Question 4

(i) Give an example of a closed and convex subset K of Euclidean space, and give an example of a continuous function $f\colon K \to K$ such that f has no fixed point.

Solution. Let $K = \mathbb{R}$. Define $f(x) = x + 1$ for any $x \in \mathbb{R}$. If $f(x) = x$, then $x = x + 1$, so that $1 = 0$, which is a contradiction. So, $f(x) \neq x$ for all $x \in \mathbb{R}$, so that f has no fixed points. As discussed in class, $\mathbb{R}$ is closed and convex. Let f is a degree one polynomial, so it is continuous.

(ii) Give an example of a bounded and closed subset K of Euclidean space, and give an example of a continuous function $f\colon K \to K$ such that f has no fixed point.

Solution. Let $C = \{(x,y) \in \mathbb{R}^2 \colon x^2 + y^2 = 1\}$. Let $f\colon C \to C$ be a clockwise rotation by $\pi/2$ radians. Then f is continuous (in fact it is linear). Also, C is closed (since it is the level set of a polynomial). But f has no fixed points, since $f(x,y) = (x,y)$ for $(x,y) \in \mathbb{R}^2$ is only satisfied by $(x,y) = (0,0)$.

(iii) Give an example of a function $f\colon \mathbb{R} \to \mathbb{R}$ such that the only fixed point of f is the point $x = 1$.

Solution. Define $f(x) = 2x - 1$ for any $x \in \mathbb{R}$. Note that $f(1) = 1$, so that $x = 1$ is a fixed point. Also, if $f(x) = x$, then $2x - 1 = x$, so $x = 1$. That is, $x = 1$ is the only fixed point of f.

5. Question 5

Let A be an $m \times n$ real matrix. Prove:

$$\min_{y \in \Delta_n} \max_{x \in \Delta_m} x^T A y = \min_{y \in \Delta_n} \max_{i=1,\ldots,m} (Ay)_i.$$

Solution. It suffices to show that $\max_{x \in \Delta_m} x^T A y = \max_{i=1,\ldots,m} (Ay)_i$. That is, it suffices to show that, for any vector $v \in \mathbb{R}^m$, we have $\max_{x \in \Delta_m} x^T v = \max_{i=1,\ldots,m} v_i$. Let $1 \leq k \leq m$ such that $v_k = \max_{i=1,\ldots,m} v_i$. Let $x = (x_1, \ldots, x_m) \in \Delta_m$. Then, for any $1 \leq i \leq m$, we have $v_i x_i \leq v_k x_i$ since $x_i \geq 0$. Therefore,

$$x^T v = \sum_{i=1}^m x_i v_i \leq \sum_{i=1}^m x_i v_k = v_k \sum_{i=1}^m x_i = v_k,$$

using that $x \in \Delta_m$, so $\sum_{i=1}^m x_i = 1$. So, taking the maximum over all $x \in \Delta_m$, we have

$$\max_{x \in \Delta_m} x^T v \leq v_k = \max_{i=1,\ldots,m} v_i.$$