Math 167, Spring 2016, UCLA — Instructor: Steven Heilman

Name: ______________ UCLA ID: ______________ Date: ______________

Signature: ______________.
(By signing here, I certify that I have taken this test while refraining from cheating.)

Final Exam

This exam contains 15 pages (including this cover page) and 10 problems. Check to see if any pages are missing. Enter all requested information on the top of this page.

You may *not* use your books, notes, or any calculator on this exam. You are required to show your work on each problem on this exam. The following rules apply:

- You have 180 minutes to complete the exam.
- **If you use a theorem or proposition from class or the notes or the book you must indicate this** and explain why the theorem may be applied. It is okay to just say, "by some theorem/proposition from class."
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this. Scratch paper is at the end of the document.

Do not write in the table to the right. Good luck![a]

Problem	Points	Score
1	15	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
Total:	105	

Reference sheet

Below are some definitions that may be relevant.

$$\Delta_m := \{x = (x_1, \ldots, x_m) \in \mathbf{R}^m \colon \sum_{i=1}^{m} x_i = 1,\ x_i \geq 0,\ \forall\, 1 \leq i \leq m\}.$$

Let m, n be positive integers. Suppose we have a two-player general sum game with $m \times n$ payoff matrices. Let A be the payoff matrix for player I and let B be the payoff matrix for player II. A pair of vectors $(\widetilde{x}, \widetilde{y})$ with $\widetilde{x} \in \Delta_m$ and $\widetilde{y} \in \Delta_n$ is a **Nash equilibrium** if

$$\widetilde{x}^T A \widetilde{y} \geq x A \widetilde{y}, \quad \forall\, x \in \Delta_m,$$

$$\widetilde{x}^T B \widetilde{y} \geq \widetilde{x} B y, \quad \forall\, y \in \Delta_n.$$

A joint distribution of strategies is an $m \times n$ matrix $z = (z_{ij})_{1 \leq i \leq m, 1 \leq j \leq n}$ such that $z_{ij} \geq 0$ for all $i \in \{1, \ldots, m\}$, $j \in \{1, \ldots, n\}$, and such that

$$\sum_{i=1}^{m} \sum_{j=1}^{n} z_{ij} = 1.$$

We say z is a **correlated equilibrium** if

$$\sum_{j=1}^{n} z_{ij} a_{ij} \geq \sum_{j=1}^{n} z_{ij} a_{kj}, \qquad \forall\, i \in \{1, \ldots, m\}, \forall\, k \in \{1, \ldots, m\}.$$

$$\sum_{i=1}^{m} z_{ij} b_{ij} \geq \sum_{i=1}^{m} z_{ij} b_{ik}, \qquad \forall\, j \in \{1, \ldots, n\}, \forall\, k \in \{1, \ldots, n\}.$$

Suppose we have a two-player symmetric game (so that the payoff matrix for player I is A, the payoff matrix for player II is B, and with $A = B^T$). Assume that A, B are $n \times n$ matrices. A mixed strategy $x \in \Delta_n$ is said to be an **evolutionarily stable strategy** if, for any pure strategy w, we have

$$w^T A x \leq x^T A x,$$

$$\text{If } w^T A x = x^T A x, \text{ then } w^T A w < x^T A w.$$

Suppose we have a game with n players together with a characteristic function $v \colon 2^{\{1,\ldots,n\}} \to \mathbf{R}$. For each $i \in \{1, \ldots, n\}$, we define the **Shapley value** $\phi_i(v) \in \mathbf{R}$ to be any set of real numbers satisfying the following four axioms:

(i) (Symmetry) If for some $i, j \in \{1, \ldots, n\}$ we have $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq \{1, \ldots, n\}$ with $i, j \notin S$, then $\phi_i(v) = \phi_j(v)$.

(ii) (No power/ no value) If for some $i \in \{1, \ldots, n\}$ we have $v(S \cup \{i\}) = v(S)$ for all $S \subseteq \{1, \ldots, n\}$, then $\phi_i(v) = 0$.

(iii) (Additivity) If u is any other characteristic function, then $\phi_i(v + u) = \phi_i(v) + \phi_i(u)$, for all $i \in \{1, \ldots, n\}$.

(iv) (Efficiency) $\sum_{i=1}^{n} \phi_i(v) = v(\{1, \ldots, n\})$.

We define a **symmetric auction**. A single object is for sale at an auction. The seller is willing to sell the object at any nonnegative price. There are n buyers, which we identify with the set $\{1, 2, \ldots, n\}$. All buyers have some set of **private values** in $[0, 1]$. We denote the private value of buyer $i \in \{1, \ldots, n\}$ by V_i, so that V_i is a random variable that takes values in $[0, 1]$. We assume that all of the random variables $V_1, \ldots, V_n$ are independent. We also assume that $V_1, \ldots, V_n$ are identically distributed, with a continuous density function. That is, there exists some continuous function $f\colon [0, 1] \to [0, \infty)$ with $\int_0^1 f(x)dx = 1$ such that: for each $i \in \{1, \ldots, n\}$, for each $t \in [0, 1]$, the probability that $V_i \leq t$ is equal to $\int_0^t f(x)dx$. We define the **expected value** of V_1 to be $\int_0^1 xf(x)dx$. Finally, we assume that all buyers are **risk-neutral**, so that each buyer seeks to maximize her expected profits.

Finally, we assume that all of the above assumptions are **common knowledge**. That is, every player knows the above assumptions; every player knows that every player knows the above assumptions; every player knows that every player knows that every player knows the above assumptions; etc.

Under the above assumptions, a **pure strategy** for Player $i \in \{1, \ldots, n\}$ is a function $\beta_i\colon [0, 1] \to [0, \infty)$. So, if Player i has a private value of V_i, he will make a bid of $\beta_i(V_i)$ in the auction. (We will not discuss mixed strategies in auctions.)

Given the strategies $\beta = (\beta_1, \ldots, \beta_n)$, and given any $v \in [0, 1]$, Player i has expected profit $P_i(\beta, v)$, if her private value is v. (If buyer i wins the auction, and if buyer i has private value v and bid b, then the profit of buyer i is $v - b$.) We say that a strategy β is an **equilibrium** if, given any $v \in [0, 1]$, any $b \geq 0$, and any $i \in \{1, \ldots, n\}$,

$$P_i(\beta, v) \geq P_i((\beta_1, \ldots, \beta_{i-1}, b, \beta_{i+1}, \ldots, \beta_n), v).$$

Let n be a positive integer. Let $x = (x_1, \ldots, x_n) \in \{-1, 1\}^n$. Let $f, g\colon \{-1, 1\}^n \to \mathbf{R}$. For any subset $S \subseteq \{1, \ldots, n\}$, define a function $W_S\colon \{-1, 1\}^n \to \mathbf{R}$ by $W_S(x) := \prod_{i \in S} x_i$. Define also the inner product $\langle f, g \rangle := 2^{-n} \sum_{x \in \{-1,1\}^n} f(x)g(x)$. Any $f\colon \{-1, 1\}^n \to \mathbf{R}$ can be expressed as $f(x) = \sum_{S \subseteq \{1, \ldots, n\}} \langle f, W_S \rangle W_S(x)$. For any $S \subseteq \{1, \ldots, n\}$, if we denote $\widehat{f}(S) := \langle f, W_S \rangle = 2^{-n} \sum_{y \in \{-1,1\}^n} f(y)W_S(y)$, then we have $f(x) = \sum_{S \subseteq \{1, \ldots, n\}} \widehat{f}(S)W_S(x)$.

The **noise stability** of f with parameter $\rho \in (-1, 1)$ is defined to be $\sum_{S \subseteq \{1, \ldots, n\}} \rho^{|S|} |\widehat{f}(S)|^2$.

1. Label the following statements as TRUE or FALSE. If the statement is true, explain your reasoning. If the statement is false, provide a counterexample and explain your reasoning. (In this question, you can freely cite results from the homeworks.)

   (a) (3 points) There is a two-person zero sum game with two Nash equilibria and one optimal strategy.

   TRUE FALSE (circle one)

   (b) (3 points) Any correlated equilibrium is a convex combination of Nash equilibria.

   TRUE FALSE (circle one)

   (c) (3 points) Every two-person zero sum game has at least one pure Nash equilibrium.

   TRUE FALSE (circle one)

   (d) (3 points) Every Evolutionarily Stable Strategy is a Nash equilibrium.

   TRUE FALSE (circle one)

   (e) (3 points) The Condorcet paradox no longer occurs if we consider an election between four candidates. That is, the Condorcet paradox only occurs in Condorcet elections between three candidates.

   TRUE FALSE (circle one)

2. (10 points) State the two-dimensional case of Sperner's Lemma. (Make sure to include all of the assumptions.) (You do **not** have to prove Sperner's Lemma.)

3. (10 points) Let $A, B \subseteq \mathbf{R}^2$ with $A \cap B = \emptyset$. We say that A, B can be *separated* if the following property holds. There exists $z \in \mathbf{R}^2$ and there exists $c \in \mathbf{R}$ such that $z^T a < c < z^T b$ for all $a \in A$ and for all $b \in B$. We say that A, B cannot be separated if it does not hold that A, B can be separated.

   Give an example of two closed, convex sets $A, B \subseteq \mathbf{R}^2$ with $A \cap B = \emptyset$, such that A, B cannot be separated. (As usual, you have to justify your answer. Also, **all** of the required conditions on A, B must be satisfied. Lastly, drawing a picture might be helpful, but it will not constitute a complete answer.)

4. (10 points) Prove that any Nash equilibrium is a Correlated Equilibrium. (That is, if m, n are positive integers, and if $(\widetilde{x}, \widetilde{y})$ is a Nash equilibrium with $\widetilde{x} \in \Delta_m$ and $\widetilde{y} \in \Delta_n$, then $\widetilde{x}\widetilde{y}^T$ is a correlated equilibrium.) (Here we regard $\widetilde{x}$ and $\widetilde{y}$ as column vectors.)

5. (10 points) Define $v\colon 2^{\{1,2,3\}} \to \mathbf{R}$ so that $v(\{1,2\}) = v(\{1,3\}) = 1$, $v(\{2,3\}) = 0$, $v(\{1,2,3\}) = 2$, and $v(\{1\}) = v(\{2\}) = v(\{3\}) = v(\emptyset) = 0$.

   Using any method you prefer, compute all of the Shapley values of v. (It is recommended you do this problem two different ways to protect against arithmetic errors.)

6. (10 points) There are five pirates on a ship. It is also common knowledge that every pirate prefers to maximize his amount of gold. There are 100 gold pieces to be split amongst the pirates. The game begins when the first pirate proposes how he thinks the gold should be split amongst the five pirates. All five pirates vote whether or not to accept the proposal, by a majority vote. If the proposal is accepted, the game ends. If the proposal is not accepted, the first pirate is thrown overboard, and the game begins continues. The second pirate now proposes how he thinks the gold should be split amongst the four remaining pirates. All four pirates vote whether or not to accept the proposal, by a majority vote (the current proposer, i.e. the second pirate breaks a tie). If the proposal is accepted, the game ends. If the proposal is not accepted, the second pirate is thrown overboard, and the game continues, etc. (During any voting phase, if a pirate's share of gold will decrease by throwing the proposer overboard, this pirate will vote to accept the proposal; otherwise this pirate will vote to not accept the proposal.) What is the largest amount of gold that the first pirate can obtain in the game?

7. (10 points) Suppose we have n buyers, and $f(v) = 1$ for any $v \in [0, 1]$ in a sealed-bid first price auction. That is, the private values $V_1, \ldots, V_n$ are uniformly distributed in the interval $[0, 1]$ (and independent). Show that an equilibrium strategy is $\beta_i(v) = \frac{n-1}{n}v$, for every $v \in [0, 1]$, for every $i \in \{1, \ldots, n\}$. (Hint: let $Z = \max(V_2, \ldots, V_n)$. Using probabilistic notation, note that $\mathbf{P}(Z \leq t) = [\mathbf{P}(V_2 \leq t)]^{n-1}$ for all $t \in \mathbf{R}$. That is, $\mathbf{P}(Z \leq t) = \int_0^t (d/ds)[\mathbf{P}(V_2 \leq s)]^{n-1} ds$.)

   (In a sealed-bid first price auction, every buyer submits a sealed envelope with her desired bid for the item. The buyer who has submitted the highest bid receives the item for their bid.)

8. (10 points)

- Let $f\colon \{-1,1\}^n \to \{-1,1\}$. Prove that the noise stability of f is at most 1.
- Let $f\colon \{-1,1\}^n \to \{-1,1\}$ with $\sum_{x \in \{-1,1\}^n} f(x) = 0$. Prove that the noise stability of f with parameter $0 < \rho < 1$ is at most ρ.

9. (10 points) Let n be a positive integer. Let $f, g\colon \{-1,1\}^n \to \{-1,1\}$. Let $a_0, \ldots, a_n, b_0, \ldots, b_n \in \mathbf{R}$. Let $x = (x_1, \ldots, x_n) \in \{-1,1\}^n$. For any $x \in \{-1,1\}^n$, define $L_f(x) = a_0 + \sum_{i=1}^n a_i x_i$, $L_g(x) = b_0 + \sum_{i=1}^n b_i x_i$. Assume that $L_f(x) \neq 0$ and $L_g(x) \neq 0$ for all $x \in \{-1,1\}^n$. Assume also that $f(x) = \mathrm{sign}(L_f(x))$ and $g(x) = \mathrm{sign}(L_g(x))$ for all $x \in \{-1,1\}^n$.

   Assume that $\widehat{f}(S) = \widehat{g}(S)$ for all $S \subseteq \{1, \ldots, n\}$ with $|S| \leq 1$. Prove that $f = g$.

10. (10 points) Explain in detail the statement of Arrow's Impossibility Theorem. (You do not need to prove the Theorem, only state it precisely, state how the Condorcet election works, etc.)

(Scratch paper)

(More scratch paper)