

541B Midterm 1 Solutions¹

1. QUESTION 1

Let $X_1, \dots, X_n$ be a random sample of size n from a Bernoulli distribution with unknown parameter $p \in [0, 1]$. (So, $\mathbf{P}(X_1 = 1) = p$ and $\mathbf{P}(X_1 = 0) = 1 - p$.)

- Find an MLE (maximum likelihood estimator) for p .
- Is the MLE you found unique? That is, could there be more than one MLE for this problem?

Solution. Suppose for now that $0 < \sum_{i=1}^n X_i < n$. Then the likelihood $\prod_{i=1}^n p^{X_i}(1-p)^{1-X_i}$ is zero when $p = 0$ or $p = 1$ and positive elsewhere, so we may assume for the purpose of maximizing the likelihood that $0 < p < 1$. The MLE of p is a value of p maximizing

$$\log \prod_{i=1}^n p^{X_i}(1-p)^{1-X_i} = \log \left((p/(1-p))^{\sum_{i=1}^n X_i} (1-p)^n \right) = (\log(p) - \log(1-p)) \sum_{i=1}^n X_i + n \log(1-p).$$

Taking a derivative in p , we get $-n/(1-p) + (1/p + 1/(1-p)) \sum_{i=1}^n X_i = 0$. Multiplying by $p(1-p)$ (which does not change the sign of the derivative of the log likelihood) gives $-np + \sum_{i=1}^n X_i = 0$. So, when $p < \frac{1}{n} \sum_{i=1}^n X_i$, the log likelihood is increasing in p , and when $p > \frac{1}{n} \sum_{i=1}^n X_i$, the log likelihood is decreasing in p . So, by the first derivative test, the unique maximum value of p occurs when $p = \frac{1}{n} \sum_{i=1}^n X_i$, so the MLE for p is $\frac{1}{n} \sum_{i=1}^n X_i$. The first derivative test shows the MLE is unique (up to probability zero changes).

In the remaining cases: if $\sum_{i=1}^n X_i = 0$, then the likelihood is $(1-p)^n$, so the unique maximum occurs at $p = 0 = \sum_{i=1}^n X_i/n$. If $\sum_{i=1}^n X_i = n$, then the likelihood is p^n , so the unique maximum occurs at $p = 1 = \sum_{i=1}^n X_i/n$.

So, in any case, the maximum likelihood exists and uniquely occurs at $\frac{1}{n} \sum_{i=1}^n X_i$.

2. QUESTION 2

Suppose X is a random variable with distribution f_θ where $\theta \in \Theta = \{\theta_0, \theta_1\}$.

If $\theta = \theta_0$, then $\mathbf{P}(X = 3) = 3/5$, $\mathbf{P}(X = 4) = 3/10$ and $\mathbf{P}(X = 5) = 1/10$.

If $\theta = \theta_1$, then $\mathbf{P}(X = 3) = 1/5$, $\mathbf{P}(X = 4) = 3/5$ and $\mathbf{P}(X = 5) = 1/5$.

We want to test the hypothesis H_0 that $\theta = \theta_0$ versus the hypothesis H_1 that $\theta = \theta_1$. Let $\mathcal{T}$ be the class of hypothesis tests with significance level at most $1/5$.

- Find a UMP class $\mathcal{T}$ test $\phi: \mathbf{R} \rightarrow [0, 1]$.
Compute all constants that appear in the definition of ϕ . Justify your answer.
- Compute the power function of ϕ .
- Is the UMP class $\mathcal{T}$ test unique? If it is unique, justify your answer. If it is not unique, find another UMP class $\mathcal{T}$ test ϕ' such that $\mathbf{P}_{\theta_0}(\phi(X) \neq \phi'(X)) > 0$.

Solution. We first compute the likelihood ratios

$$\frac{f_{\theta_1}(3)}{f_{\theta_0}(3)} = \frac{1}{3}, \quad \frac{f_{\theta_1}(4)}{f_{\theta_0}(4)} = 2, \quad \frac{f_{\theta_1}(5)}{f_{\theta_0}(5)} = 2.$$

The Neyman-Pearson Lemma says that a UMP test for the class of tests with an upper bound on the significance level must be a likelihood ratio test with significance level equal

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to $1/5$. That is, there is some $k > 0$ and $\gamma \in [0, 1]$ such that the following hypothesis test is UMP class $\mathcal{T}$.

$$\phi(x) := \begin{cases} 1 & , \text{ if } f_{\theta_1}(x) > k f_{\theta_0}(x) \\ 0 & , \text{ if } f_{\theta_1}(x) < k f_{\theta_0}(x) \\ \gamma & , \text{ if } f_{\theta_1}(x) = k f_{\theta_0}(x). \end{cases}$$

After examining the likelihood ratios, we decide to choose $k = 2$, so that

$$\phi(x) := \begin{cases} 1 & , \text{ if } f_{\theta_1}(x) > 2 f_{\theta_0}(x) \\ 0 & , \text{ if } f_{\theta_1}(x) < 2 f_{\theta_0}(x) \\ \gamma & , \text{ if } f_{\theta_1}(x) = 2 f_{\theta_0}(x) \end{cases} = \begin{cases} 0 & , \text{ if } x \notin \{4, 5\} \\ \gamma & , \text{ if } x = 4, 5. \end{cases}$$

Then $\mathbf{E}_{\theta_0}\phi(X) = \mathbf{P}_{\theta_0}(X = 4, 5)\gamma = (3/10 + 1/10)\gamma = (2/5)\gamma$. Since this quantity is equal to $1/5$ by assumption, we choose $\gamma := 1/2$. That is, a UMP test is

$$\phi(x) := \begin{cases} 0 & , \text{ if } x \notin \{4, 5\} \\ 1/2 & , \text{ if } x = 4, 5. \end{cases}$$

The power function of ϕ is $\beta(\theta) := \sup_{\theta \in \Theta_0} \mathbf{E}_{\theta}\phi(X) = \mathbf{E}_{\theta_0}\phi(X)$ which we described above.

Finally, the UMP class $\mathcal{T}$ test is not unique. For example, if $\phi'(4) = a$ and $\phi'(5) = b$ with $\phi' = 0$ otherwise, then we require $1/5 = \mathbf{E}_{\theta_0}\phi'(X) = (3/10)a + (1/10)b$, so that $3a + b = 2$ and $0 \leq a, b \leq 1$. Above we chose $a = b = 1/2$, but we could also e.g. choose $b = 0$ and $a = 2/3$. Then ϕ' is UMP by the Neyman-Pearson Lemma (it is still a likelihood ratio test, with significance level $1/5$), but $\mathbf{P}_{\theta_0}(\phi(X) \neq \phi'(X)) \geq \mathbf{P}_{\theta_0}(X = 5) = 1/10 > 0$, since $\phi(5) = 1/2 \neq 0 = \phi'(5)$.

3. QUESTION 3

Let $X_1, \dots, X_n$ be a real-valued random sample of size n so that X_1 has PDF given by

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-(x-\mu)^2/2}, \quad \forall x \in \mathbf{R},$$

where $\mu \in \mathbf{R}$ is an unknown parameter.

- Suppose we want to test the hypothesis H_0 that $\mu = 0$ versus the hypothesis H_1 that $\mu > 0$. Describe a uniformly most powerful hypothesis test among all hypothesis tests with significance level at most $1/2$. Justify your answer.
- Suppose we want to test the hypothesis H_0 that $\mu = 0$ versus the hypothesis H_1 that $\mu \neq 0, \mu \in \mathbf{R}$. Does there exist a uniformly most powerful hypothesis test among all hypothesis tests with significance level at most $1/2$? Justify your answer.

Solution. We have for any $x = (x_1, \dots, x_n) \in \mathbf{R}^n$,

$$f_{\mu}(x) = \prod_{i=1}^n \frac{1}{\sqrt{2\pi}} e^{-(x_i - \mu)^2/2} = (2\pi)^{-n/2} e^{-\sum_{i=1}^n (x_i - \mu)^2}.$$

So, if $\mu_1 > \mu_0 > 0$, we have for any $x = (x_1, \dots, x_n) \in \mathbf{R}^n$,

$$\frac{f_{\mu_1}(x)}{f_{\mu_0}(x)} = e^{-\sum_{i=1}^n [(x_i - \mu_1)^2 - (x_i - \mu_0)^2]/2} = e^{\sum_{i=1}^n [2x_i(\mu_1 - \mu_0) - (\mu_1^2 - \mu_0^2)]/2}.$$

Since $\mu_1 - \mu_0 > 0$, this likelihood ratio is a strictly increasing function of $\sum_{i=1}^n x_i$. We conclude from the Karlin-Rubin Theorem (with $\Theta = [0, \infty)$) that there is a UMP test is of the form

$$\phi(x) := \begin{cases} 1 & , \text{ if } \sum_{i=1}^n x_i > c \\ 0 & , \text{ if } \sum_{i=1}^n x_i < c \\ \gamma & , \text{ if } \sum_{i=1}^n x_i = c. \end{cases}$$

for some $c \in \mathbf{R}$. Rewriting this, and noting that $\sum_{i=1}^n x_i = c$ has probability zero for any $\mu \in \mathbf{R}$,

$$\phi(x) := \begin{cases} 1 & , \text{ if } \sum_{i=1}^n x_i > c \\ 0 & , \text{ if } \sum_{i=1}^n x_i \leq c. \end{cases}$$

To obtain significance level 1/2, we may choose $c := 0$.

Finally: no, the UMP does not exist when H_1 is $\mu \neq 0, \mu \in \mathbf{R}$. If such a ϕ' did exist, then from the first part of the problem we would have $\phi = \phi'$. (Karlin-Rubin does not guarantee uniqueness of the UMP, but re-doing the proof, i.e. applying the Neyman-Pearson Lemma, demonstrates uniqueness in this particular example.) But then ϕ' must also be UMP for testing $\mu = 0$ versus $\mu < 0$, so (e.g. changing $X_1, \dots, X_n$ to $-X_1, \dots, -X_n$ and re-doing the previous argument), ϕ' must reject when $\sum_{i=1}^n x_i < 0$, so that $\phi' = 1$ for all x (with probability one), i.e. $\mathbf{E}_0 \phi'(X) = 1$, a contradiction.

4. QUESTION 4

Suppose $\Theta = \{\theta_0, \theta_1\}$, $\Theta_0 = \{\theta_0\}$, $\Theta_1 = \{\theta_1\}$. Let H_0 be the hypothesis $\{\theta = \theta_0\}$ and let H_1 be the hypothesis $\{\theta = \theta_1\}$. Let $\{f_{\theta_0}, f_{\theta_1}\}$ be two multivariable probability densities on $\mathbf{R}^n$. Fix $k \geq 0$. Define a **likelihood ratio test** $\phi: \mathbf{R}^n \rightarrow [0, 1]$ to be

$$\phi(x) := \begin{cases} 1 & , \text{ if } f_{\theta_1}(x) > k f_{\theta_0}(x) \\ 0 & , \text{ if } f_{\theta_1}(x) < k f_{\theta_0}(x) \\ (\text{unspecified}) & , \text{ if } f_{\theta_1}(x) = k f_{\theta_0}(x). \end{cases} \quad (*)$$

Define

$$\alpha := \sup_{\theta \in \Theta_0} \beta(\theta) = \beta(\theta_0) = \mathbf{E}_{\theta_0} \phi(X). \quad (**)$$

Let $\mathcal{T}$ be the class of all randomized hypothesis tests with significance level at most α .

Prove the following:

Any randomized hypothesis test satisfying (*) is a UMP class $\mathcal{T}$ test.

[This exercise asks you to prove the first part of the Neyman-Pearson Lemma.]

Solution. As we already noted in (**), Θ_0 consists of a single point, so the supremum appearing in (**) is just $\beta(\theta_0)$, and we will repeatedly use this fact below without further mention.

Let $\beta(\theta)$ be the power function of the test corresponding to ϕ . Let ϕ' another test in $\mathcal{T}$, and let $\beta'(\theta)$ be the power function of this test. By definition of ϕ , we have

$$[\phi(x) - \phi'(x)][f_{\theta_1}(x) - k f_{\theta_0}(x)] \geq 0, \quad \forall x \in \mathbf{R}^n.$$

Therefore,

$$0 \leq \int_{\mathbf{R}^n} [\phi(x) - \phi'(x)][f_{\theta_1}(x) - k f_{\theta_0}(x)] dx = \beta(\theta_1) - \beta'(\theta_1) - k[\beta(\theta_0) - \beta'(\theta_0)]. \quad (***)$$

Since ϕ has significance level α and ϕ' has significance level at most α , we have $\beta(\theta_0) - \beta'(\theta_0) \geq 0$. So, $k \geq 0$ and $(***)$ imply that $\beta(\theta_1) - \beta'(\theta_1) \geq 0$. That is, the ϕ test is UMP class $\mathcal{T}$.