

Name: _____ USC ID: _____ Date: _____

Signature: _____. Discussion Section: _____

(By signing here, I certify that I have taken this test while refraining from cheating.)

Exam 1

This exam contains 6 pages (including this cover page) and 4 problems. Enter all requested information on the top of this page.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- You have 50 minutes to complete the exam, starting at the beginning of class.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit.** A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this. Scratch paper appears at the end of the document.

Problem	Points	Score
1	8	
2	12	
3	12	
4	10	
Total:	42	

Do not write in the table to the right. Good luck!^a

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1. (8 points) Let $X_1, \dots, X_n$ be a random sample of size n from a Bernoulli distribution with unknown parameter $p \in [0, 1]$. (So, $\mathbf{P}(X_1 = 1) = p$ and $\mathbf{P}(X_1 = 0) = 1 - p$.)
- Find an MLE (maximum likelihood estimator) for p .
 - Is the MLE you found unique? That is, could there be more than one MLE for this problem?

[this was a modified homework/practice exam question]

2. (12 points) Suppose X is a random variable with distribution f_θ where $\theta \in \Theta = \{\theta_0, \theta_1\}$.

If $\theta = \theta_0$, then $\mathbf{P}(X = 3) = 3/5$, $\mathbf{P}(X = 4) = 3/10$ and $\mathbf{P}(X = 5) = 1/10$.

If $\theta = \theta_1$, then $\mathbf{P}(X = 3) = 1/5$, $\mathbf{P}(X = 4) = 3/5$ and $\mathbf{P}(X = 5) = 1/5$.

We want to test the hypothesis H_0 that $\theta = \theta_0$ versus the hypothesis H_1 that $\theta = \theta_1$.

Let $\mathcal{T}$ be the class of hypothesis tests with significance level at most $1/5$.

- Find a UMP class $\mathcal{T}$ test $\phi: \mathbf{R} \rightarrow [0, 1]$.
Compute all constants that appear in the definition of ϕ . Justify your answer.
- Compute the power function of ϕ .
- Is the UMP class $\mathcal{T}$ test unique? If it is unique, justify your answer. If it is not unique, find another UMP class $\mathcal{T}$ test ϕ' such that $\mathbf{P}_{\theta_0}(\phi(X) \neq \phi'(X)) > 0$.

[this was a modified homework/practice exam question]

3. (12 points) Let $X_1, \dots, X_n$ be a real-valued random sample of size n so that X_1 has PDF given by

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-(x-\mu)^2/2}, \quad \forall x \in \mathbf{R},$$

where $\mu \in \mathbf{R}$ is an unknown parameter.

- Suppose we want to test the hypothesis H_0 that $\mu = 0$ versus the hypothesis H_1 that $\mu > 0$. Describe a uniformly most powerful hypothesis test among all hypothesis tests with significance level at most $1/2$. Justify your answer.
- Suppose we want to test the hypothesis H_0 that $\mu = 0$ versus the hypothesis H_1 that $\mu \neq 0, \mu \in \mathbf{R}$. Does there exist a uniformly most powerful hypothesis test among all hypothesis tests with significance level at most $1/2$? Justify your answer.

[this was a modified homework/practice exam question]

4. (10 points) Suppose $\Theta = \{\theta_0, \theta_1\}$, $\Theta_0 = \{\theta_0\}$, $\Theta_1 = \{\theta_1\}$. Let H_0 be the hypothesis $\{\theta = \theta_0\}$ and let H_1 be the hypothesis $\{\theta = \theta_1\}$. Let $\{f_{\theta_0}, f_{\theta_1}\}$ be two multivariable probability densities on $\mathbf{R}^n$. Fix $k \geq 0$. Define a **likelihood ratio test** $\phi: \mathbf{R}^n \rightarrow [0, 1]$ to be

$$\phi(x) := \begin{cases} 1 & , \text{ if } f_{\theta_1}(x) > k f_{\theta_0}(x) \\ 0 & , \text{ if } f_{\theta_1}(x) < k f_{\theta_0}(x) \\ (\text{unspecified}) & , \text{ if } f_{\theta_1}(x) = k f_{\theta_0}(x). \end{cases} \quad (*)$$

Define

$$\alpha := \sup_{\theta \in \Theta_0} \beta(\theta) = \beta(\theta_0) = \mathbf{E}_{\theta_0} \phi(X). \quad (**)$$

Let $\mathcal{T}$ be the class of all randomized hypothesis tests with significance level at most α .

Prove the following:

Any randomized hypothesis test satisfying (*) is a UMP class $\mathcal{T}$ test.

[This exercise asks you to prove the first part of the Neyman-Pearson Lemma.]

[this was a practice exam question]

(Scratch paper)